Bank management via stochastic optimal control

Janine Mukuddem-Petersen, Mark A. Petersen

Department of Mathematics and Applied Mathematics, North-West University, Potchefstroom 2520, South Africa

Received 12 March 2005; received in revised form 9 January 2006; accepted 20 March 2006

Abstract

This paper examines a problem related to the optimal risk management of banks in a stochastic dynamic setting. In particular, we minimize market and capital adequacy risk that involves the safety of the securities held and the stability of sources of funds, respectively. In this regard, we suggest an optimal portfolio choice and rate of bank capital inflow that will keep the loan level as close as possible to an actuarially determined reference process. This set-up leads to a nonlinear stochastic optimal control problem whose solution may be determined by means of the dynamic programming algorithm. The above analysis is reliant on the construction of continuous-time stochastic models for bank behaviour upon which a spread method for loan capitalization is imposed.

Keywords: Dynamic programming; Stochastic optimal control; Financial institutions: Banks; Finance: Management

1. Introduction

Bank management mainly focusses on four operational concerns. Firstly, the bank has to be liquid enough to finance its obligations to depositors. This aspect of bank management is called liquidity management and involves the acquisition of sufficiently liquid assets to meet the demand from deposit withdrawals and depositor payments. In addition, banks engage in liability management that entails the sourcing of funds at an acceptable cost. Thirdly, banks must have incentives to invest in assets that have a reasonably low level of risk associated with them. This process is known as asset management and aims to encourage investment in assets that have a low default probability and strategies that are sufficiently diverse. Finally, capital adequacy management involves the decision about the amount of capital the bank should hold and how it should be accessed. Our contribution mainly addresses problems associated with the latter two aspects of bank management.

In modern times, the volume of regulation drafted for the banking industry has increased dramatically. In this regard, the Basel II capital accord (see, for instance, Basel II (Basel Committee on Banking Supervision, September 2001, June 2004)) places a major emphasis on the regulatory management of banks. In particular, we note that the June 2004 revised framework of this accord encourages improved risk management strategies in the banking sector (see Basel II, June 2004). The main categories of risks that banks bear are credit, market, operational and liquidity risks. Our paper discusses optimal behaviour of a bank with respect to equity and capital adequacy risk. Equity risk (subcategory of market risk impacting asset management) is the risk of losses arising from movements in the equity market. Capital adequacy risk (subcategory of liquidity risk impacting capital adequacy management) refers to the risk that is associated with capital flows from share- and debtholders, retained earnings and loan-loss reserves that are collectively known as bank capital. In our contribution, equity and capital adequacy risks are minimized by optimal marketable security allocation and capital inflow, respectively. The study of the dynamics of these risk minimization strategies has always been an important issue in the management of banks. In particular, Dangl and Lehar (2004) and Decamps, Rochet, and Roger (2004), construct continuous-time models which permit optimal control problems to be solved in the context of portfolio selection and capital requirements (see also, Fouche, Gideon, Mukuddem-Petersen, and Petersen, 2006a, 2006b).
2. Stochastic model of a bank

Banks operate via the process of asset transformation which entails the selling of liabilities with certain properties (combination of liquidity, risk, size and return) and using the proceeds to buy assets with other characteristics. In this regard, we consider a continuous-time dynamic model in which the bank holds assets (uses of funds) and has liabilities (sources of funds) that behave in a stochastic manner (see Petersen, Raubenheimer, and van der Walt, 2005 for an analogue in insurance theory). This behaviour is consistent with the uncertainty associated with items appearing on the balance sheet, namely, the reserves, loans and securities (assets) and deposits and borrowings (liabilities). The aforementioned items are balanced by the bank capital according to the well-known relation

Total assets = Total liabilities + Bank capital.

In this regard, a stylized balance sheet of a typical bank at time \( t \) can be represented as

\[
R(t) + L(t) + S(t) = D(t) + B(t) + C(t),
\]

where we have that

\[
R : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Reserves},
L : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Loans},
S : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Securities},
D : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Deposits},
B : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Borrowings},
C : \Omega \times T \rightarrow \mathbb{R}_+ : -\text{Bank capital}.
\]

2.1. Bank assets

The main issues discussed in this subsection are bank loans and securities. At the outset, we assume that we are working with a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) on a time period \( T = [0, t_f] \). Here we have that \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) is a complete, right continuous filtration generated by the one-dimensional Brownian motion \( \{W_t(t)\}_{t \geq 0} \). Also, \( \mathbb{P} \) is a probability measure on \( \Omega \).

2.1.1. Reserves

All banks hold some of the funds they acquire as deposits in an account at the federal reserve. Reserves, \( R \), are those deposits plus currency that is physically held by banks. The two reasons that reserves are held is because there may be some reserve requirements (in the main these have been replaced by capital requirements à la Basel II) in place and the fact that they are the most liquid of all bank assets that can be used to meet obligations when funds are withdrawn.
2.1.2. Loans

In the sequel, the stochastic process \( l : \Omega \times T \rightarrow \mathbb{R} \) is the **lending rate** and is defined as the net rate resulting from loan issuing, amortizations and defaults. \( l \) is random and we choose to represent it by means of geometric Brownian motion which makes the problem more analytically tractable. Thus, in some ways, the lending rate will reflect reality by, for instance, having positive values. Also, in this case, increments will follow lognormal distributions. It is generally acceptable to model \( l \) as

\[
dl(t) = l(t)[r(t) dt + \sigma_l dW_l(t)], \quad l_0 = l(0),
\]

where \( r(t) \in \mathbb{R} \) and \( \sigma_l \in \mathbb{R}_+ \) is the volatility in the lending rate with the differential \( dW_l(t) \) the increment of the Brownian motion \( W_l(t) \) representing the random shocks loan may be exposed to. In the main, the lending rate, \( l \), is dependent on the level of macroeconomic activity in the bank’s loan market. Furthermore, \( l \), which solves (2), may be expressed as

\[
l(t) = l_0 \exp \left( \left( r - \frac{\sigma_l^2}{2} \right) t + \sigma_l W_l(t) \right).
\]

In principle, this means that the change in the lending rate occurs at a constant exponential rate.

2.1.3. Securities

In the sequel, we denote the value of bank securities by \( y_i(t) \) and \( y_i(0) = 1 \). (4)

where \( r \) is a deterministic rate of return. Also, the evolution of the marketable securities is given by

\[
dy_i(t) = y_i(t) \left[ b_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right], \quad y_i(0) = y_0,
\]

\[1 \leq i \leq n.\]

Here \( b_i \) and \( \sigma_{ij}, 1 \leq i, j \leq n \) are positive constants and the vector

\[(W_0(t), W_1(t), \ldots, W_n(t))^T\]

is an \( n + 1 \)-dimensional Brownian motion defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \{\mathcal{F}_t\}_{t \geq 0} \) represents the completion of the filtration

\[
\sigma([W_0(t), W_1(t), \ldots, W_n(t)]^T : 0 \leq s \leq t).
\]

Given that the lending rate, \( l(t) \), is contingent on the increase in the returns on securities, we assume the existence of correlation \( q_i \in [-1, 1] \) between the Brownian motions \( W_i \) and \( W_j \) for \( i = 1, \ldots, n \). It follows that

\[
E(W_i(t)W_j(s)) = q_{ij} \min(t, s)
\]

for \( i = 1, \ldots, n \) and \( W_i(t) = \sqrt{1 - q_i^2} W_0(t) + q_i W_i(t) \), where

When \( q_i^T q \neq 1 \) the risk in loan issuing cannot be disposed of by trading in the financial market.

Subsequently, the **market price of risk**, \( \tilde{\zeta} \), is given by

\[
\tilde{\zeta} = \tilde{b} - \rho^T \tilde{\zeta},
\]

where \( \tilde{b} = (b_1, \ldots, b_n)^T \), \( \rho \) is a column vector of 1’s and the matrix \( \tilde{\zeta} \) is assumed to be invertible. We recall that the market price of risk is the measure of the extra return, or risk premium, that investors in a company demand to bear risk due to investment in marketable securities. It is usually described by the **reward-to-risk ratio** of the market portfolio. Moreover, in the situation described above, we can compute the risk premium, \( \rho_i \), on marketable security \( i \) as

\[
\rho_i = \sum_{j=1}^{n} \sigma_{ij} \tilde{\zeta}_j,
\]

where \( \tilde{\zeta} = (\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n)^T \). From (7) and (8) we can deduce that

\[
b_i = r + \sum_{j=1}^{n} \sigma_{ij} \tilde{\zeta}_j.
\]

(9) reflects the fact that returns from marketable securities are generally higher than returns from treasuries so that it is realistic to have \( b_i > r \) for each \( i = 1, 2, \ldots, n \). In addition, the SDE (5) will become

\[
dy_i(t) = y_i(t) \left[ \left( r + \sum_{j=1}^{n} \sigma_{ij} \tilde{\zeta}_j \right) dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) \right],
\]

\[1 \leq i \leq n.

Here we note that \( y_i \) is the total return (amount of a single investment in security \( i \) with reinvestment of dividend income) of an investment made exclusively in security \( i \).

In the discussion that follows, \( \sigma \) is the symbol used to denote the invertible matrix \( (\sigma_{ij}) \) which results in the symmetric matrix
$\chi = \sigma \sigma^T$ being positive definite. Furthermore, $\pi_i(t)$ denotes the investment in the $i$th securities class at time $t$, for $0 \leq i \leq n$. In this case, the amount invested in the treasuries is given by

$$S = \sum_{i=1}^{n} \pi_i.$$  

Finally, we have that the portfolio process or trading strategy $\tilde{\pi}(t)$ is given by $\tilde{\pi}(t) = (\pi_1(t), \ldots, \pi_n(t))^T$. Here, $\tilde{\pi}(t)$ is taken to be a $\mathbb{R}^n$-measurable process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ such that

$$\int_0^\infty \tilde{\pi}(s) \tilde{\pi}(s) \, ds < \infty, \quad \text{a.s.} \quad (10)$$

2.2. Bank liabilities

Liabilities constitute the sources of funds for banks. These funds are used to purchase income-earning assets. The dynamics of the bank's liabilities is stochastic because its value has a reliance on, for instance, deposits with randomness associated with it.

2.2.1. Deposits

The majority of a bank’s liabilities consists of retail deposits, $D$, which are fully insured by a deposit insurance fund (DIF). Next, we may apply a simple banking principle (see, for instance, p. 207 of Chapter 9 in Mishkin, 2004) that relates the deposits, $D$, to the reserves, $R$, by the formula

$$R(t) = \varepsilon D(t), \quad (11)$$

where $\varepsilon$ is real-valued. From (1) and (11) we may conclude that

$$L(t) + S(t) = (1 - \varepsilon)D(t) + B(t) + C(t). \quad (12)$$

2.2.2. Borrowings

In the sequel, $B : \Omega \times T \to \mathbb{R}$ denotes borrowing from other banks and the federal reserve whose value at time $t$ is denoted by $B(t)$. Although other factors may come into play, it is an indisputable fact that the evolution of borrowings and bank assets are closely related. How this relationship can be quantified is not fully understood but largely dependent on the investment philosophy of the particular bank. For sake of argument, in the sequel, we assume that a change (increase or decrease) in borrowings from other banks result in an equal change (increase or decrease) in securities according to the rule

$$dB(t) = \sum_{i=1}^{n} \pi_i(t) \frac{dy_i(t)}{y_i(t)} + \left(S(t) - \sum_{i=1}^{n} \pi_i(t)\right) \frac{dy_0(t)}{y_0(t)}. \quad (13)$$

In this regard, it is possible in an environment of falling interest rates that share- and debtholders may be switching out of bank deposits into equities/long term bonds. This may necessitate that banks substitute deposits with interbank borrowing, while the bank is making very good returns on its liquid portfolio.

2.3. Bank capital

A bank’s capital, $C$, has the form

$$C(t) = C_{T1}(t) + C_{T2}(t) + C_{T3}(t),$$

where $C_{T1}(t)$, $C_{T2}(t)$, and $C_{T3}(t)$ are Tier 1, Tier 2 and Tier 3 capital, respectively. Roughly speaking Tier 1 (T1) capital is the book value of the bank’s stock or equity held by shareholders plus retained earnings. Also, Tier 2 (T2) and Tier 3 (T3) capital (collectively known as supplementary capital) is the sum of loan-loss reserves and subordinate debt held by debtholders. The dynamics of the bank capital can be represented as a diffusion process (see, for instance, Decamps et al., 2004; Hancock, Laing, and Wilcox, 2004; Leland, 1994) in the form

$$dC(t) = c(t) \, dt + \sigma_c \, dW_c(t), \quad C_0 = C(t_0), \quad (14)$$

where $c$ is the bank capital contribution rate and $C(t_0) = C_0$. In reality, $c$ may depend on such factors as profit flow, asset substitution and transaction costs. Subsequently, we assume that $c$ is a measurable adapted process with respect to the filtration $\{\mathcal{F}_t\}$ that satisfies

$$\int_0^\infty |c(s)| \, ds < \infty, \quad \text{a.s.} \quad (15)$$

2.4. Description of the banking model

From (12), we can deduce that the dynamics of the stylized balance sheet will be

$$dL(t) + dS(t) = (1 - \varepsilon) dD(t) + dB(t) + dC(t). \quad (16)$$

In addition, if we set $\varepsilon = 1$ in (16), then

$$dS(t) = dB(t) + dC(t) - dL(t). \quad (17)$$

In subsequent discussions, we consider the case where $\sigma_\varepsilon = 0$ in (14). This choice and consideration of the expressions for $dL$ and $dB$ given by (2) and (13), respectively, enables us to re-write the dynamics of $S$ from (17) in the form

$$dS(t) = \sum_{i=1}^{n} \pi_i(t) \frac{dy_i(t)}{y_i(t)} + \left(S(t) - \sum_{i=1}^{n} \pi_i(t)\right) \frac{dy_0(t)}{y_0(t)} + [c(t) - l(t)] \, dt. \quad (18)$$

2.5. Reference processes for banks

Actuarial cost methods involve valuation techniques used to determine the proper charges against annual operating techniques and to measure the required asset and liability levels of a financial institution at any given date. As far as we know, the actuarial cost method has not been used to discuss stipulated levels of loans issuing in the banking sector before. For the individual cost method, the unfunded loans, $l_u$, is given by the difference between the bank’s LI reference process, $l$, and the securities available for funding these loans, $S$. Symbolically, we have that

$$l_u(t) = l(t) - S(t). \quad (19)$$
These methods involve the bank’s LI reference process, \( l_t \), and the BC reference process, \( c_t \), which is then adjusted to deal with \( l_u \). This adjustment is achieved via additional bank capital, \( c_a \), for which a number of choices can be made. In this case, we have that

\[
c_a(t) = c(t) - c_t(t).
\]

To what extent shareholders are willing to make additional contributions to the equity component of the bank capital inflow of a bank that may not be operating optimally or that is in danger of insolvency is always a thorny issue. However, we make the assumption that the decision-making of the shareholders and management are perfectly aligned.

In this subsection, we consider the reference processes that will serve as a guide to establishing an ideal level of loan issuing at the end of the risk horizon. Our analysis will involve the time instances \( t_0 \) that will represent the beginning of the risk horizon, \( t_1 \) that will represent the end of the risk horizon and \( v \) that will be an arbitrary time instant. In the sequel, we denote the discount rate of interest associated with the valuation of the bank by \( r_d \). In our context, this term is used to describe the annual growth rate of an investment, used when a future value is assumed and you are trying to find the required present value.

### 2.5.1. Description of the reference processes

Throughout the discussion in this section, \( \mathbb{E}(\cdot | \mathcal{F}_t) \) denotes conditional expectation with respect to the filtration associated with the standard Brownian motion \( \{W(t)\}_t \geq 0 \). The LI reference process describes the value, using actuarial methods and assumptions, placed on the obligations of a bank to issue loans. Also, it pertains to the present value of future bank outflows resulting from loan issuing. The BC reference process, is the actuarially determined amount of bank capital that is intended to finance for some specified period time a stipulated level of loans issuing. A bank’s BC reference process is an indication of the level of incoming bank capital if the uses of funding is exactly on target and the accrued LI process is exactly covered by the securities accumulated by the bank.

Explicit formulas for the LI and BC reference processes are now provided. For every \( t > 0 \), we assume that \( l_t \) can be represented by the formula

\[
l_t(t) = \int_{t_0}^{t_1} \exp[-r_d(t_1 - v)] P(v) \times \mathbb{E}(l(t + t_1 - v) | \mathcal{F}_t) \, dv, \quad P(t_0) = 0.
\]

The capital accumulated by the bank during a certain loan capitalization period is distributed in accordance with the distribution function, \( P \), with an accompanying density function, \( p \) (compare, for instance, Chapter 3 of Bowers, Gerber, Hickman, Jones, and Nesbitt, 1997). Thus for a certain time instant, \( v \), occurring during this period the value \( P(v) \) represents the percentage of the value of the target loans to be issued accumulated up to and including \( v \). Furthermore, for every \( t > 0 \), we suppose that the BC reference process, \( c_t \), may be given by

\[
c_t(t) = \int_{t_0}^{t_1} \exp[-r_d(t_1 - v)] p(v) \times \mathbb{E}(l(t + t_1 - v) | \mathcal{F}_t) \, dv, \quad p(t_0) = 0.
\]

We note that for \( v \leq t_0 \) or \( v \geq t_1 \) we have that \( p(v) = 0 \) since the support of \( p \) is the fixed interval \([t_0, t_1] \). Also, we have that

\[
p’ = p.
\]

### 2.5.2. Relationships between reference processes

In the sequel, \( r_a \) will denote the rate of actualization where actualization is understood to mean the present value of a stipulated future process volume. A high rate of actualization suggests that the bank’s concern is more with the present than the distant future. The relationship between the reference processes \( l_t \) and \( c_t \) given by (20) and (21), respectively, and the stipulated loans, \( l \), is elucidated by the following result.

**Theorem 2** (Relationship between \( l_t, c_t \) and \( l \)). In the case where (2) holds, there exist constants \( K \) and \( L \) such that

\[
c_t = Kl \quad \text{and} \quad l_t = Li.
\]

Moreover, the following identities hold.

\[
K = 1 + (r_l - r_a) L
\]

and

\[
(r_a - r_l)l_t(t) + c_t(t) - l(t) = 0
\]

holds for every \( t \geq 0 \). If \( L \) and \( K \) are real-valued, non-zero constants then

\[
c_t(t) = Jl_t(t) \quad \text{and} \quad c_a(t) = J[l_a(t) + S(t)] - l(t),
\]

for all \( t \),

where \( J = L/K \) is a valuation rate of interest given, in terms of \( K, r_l \) and \( r_a \), by

\[
J = \frac{1}{K} + r_l - r_a.
\]

Also, we have that

\[
dl_t(t) = l_t(t)[r_l \, dt + \sigma_l \, dW_l(t)],
\]

where \( l_t(0) = Ll_0 \).

**Proof.** Since \( l \) is modelled as geometric Brownian motion and (3) holds, it follows that for every \( v \in [t_0, t_1] \) and \( t \geq 0 \), we have

\[
\mathbb{E}(l(t + t_1 - v) | \mathcal{F}_t)
\]

\[
= l_0 \exp[r_l(t_1 - v)] \exp \left\{ \left( r_l - \frac{\sigma_l^2}{2} \right) t + \sigma_l W_l(t) \right\}
\]

\[
= \exp[r_l(t_1 - v)] l(t).
\]

In order to complete the proof of the first part of the theorem we have to define the LI and BC reference process constants by

\[
L = \int_{t_0}^{t_1} \exp[(r_l - r_a)(t_1 - v)] P(v) \, dv
\]
and
\[ K = \int_{t_0}^{t_1} \exp\{(r_l - r_a)(t_1 - v)\} P(v) \, dv, \] (26)
respectively.

To show that (23) holds we bear (22) in mind and integrate by parts in the integral (26) in order to obtain
\[ K = \exp\{(r_l - r_a)(t_1 - v)\} P(v) \exp\{(r_l - r_a)(t_1 - v)\} + (r_l - r_a) \]

The last equality follows from the fact that \( P(t_0) = 0 \) as stated in (20). The fact that (24) holds follows from the first part of the proof of this theorem and a consideration of the formulas (2) and (21). The rest of the proof is easily deduced from the discussion above. \( \square \)

3. Optimal risk management of banks

In this section, we state and prove our main result (see Theorem 6) that elucidates the role of stochastic optimal control in the management of capital and market risk. A measure of the capital adequacy risk is the size of the deviations of the bank capital inflow rate from the BC reference process and is associated with the stability of sources of funds for the bank. On the other hand, the market risk is measured by the volume of the unfunded loans with its value being an indication of the security of the bank. The objective of management is to control the stability and security of the bank by minimizing a convex combination of the two aforementioned types of risk. Prior to this we discuss the stochastic dynamics of bank securities, the formulation of the optimal control problem and the spread method for bank loan capitalization.

3.1. Stochastic dynamics of the portfolio

By making use of (4), (5) and (18), we find that the dynamics of bank securities may be represented by the SDE
\[ dS(t) = \sum_{i=1}^{n} \pi_i(t) \left( b_i \int_t^{l(t)} \, dt + \sum_{j=1}^{n} \sigma_{ij} \, dW_j(t) \right) \]
\[ + \left( c(t) - l(t) \right) \, dt + \left( S(t) - \sum_{i=1}^{n} \pi_i(t) \right) r \, dt \]
\[ = \left( r S(t) + \sum_{i=1}^{n} \pi_i(t)(b_i - r) \right) \, dt \]
\[ + \left( c(t) - l(t) \right) \, dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) \sigma_{ij} \, dW_j(t), \] (27)

with initial condition \( S(0) = S_0 \). By applying Theorem 2 we can rewrite (27) as
\[ dS(t) = \left( r S(t) + \sum_{i=1}^{n} \pi_i(t)(b_i - r) \right) \, dt \]
\[ + \left( c_a(t) - (r_l - r_a)l(t) \right) \, dt \]
\[ + \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) \sigma_{ij} \, dW_j(t). \] (28)

Here the additional bank capital, \( c_a \), is chosen with the aim of elevating the expected value of the unfunded loans to zero. For a bank, a means of accomplishing this is via capital from shareholders and debtholders, retained earnings and loan-loss reserves. We note that the model describes an incomplete market because the stipulated loans, \( l \), cannot be traded in the security market and, therefore, the bank cannot hedge the inherent credit risk. Moreover, the state variables in the system (28) can be identified as the value of the securities available for investment, \( S \), and the LI reference process, \( l_t \). Also, it is clear that the control variables are the security allocation, \( \pi \), and additional bank capital, \( c_a \).

3.2. The stochastic optimal control problem

In order for a bank to determine an optimal bank capital inflow rate, \( c^* \), and security allocation strategy, \( \pi^* \), it is imperative that a well-defined value function, \( V \), with appropriate constraints is considered. The choice has to be carefully made in order to avoid ambiguous solutions to our stochastic control problem. In this particular contribution, we choose to determine a control law, \( g \), that minimizes the value function \( V : \mathcal{C}_{S_0,l_0} \to \mathbb{R}^+ \), where \( \mathcal{C}_{S_0,l_0} \) is the class of admissible control laws
\[ \mathcal{C}_{S_0,l_0} = \{ \text{Markovian stationary} (c_a(t), \tilde{\pi}(t)) \text{ adapted to} \} \]
\[ [\mathcal{C}_t]_{t \geq 0} \text{ satisfying (10) and (15) with (25) and (28) admitting a } \mathcal{G}_t \text{-measurable unique solution with continuous paths} \}. \] (29)

Here the closed-loop system for \( (c_a(t), \tilde{\pi}(t)) \in \mathcal{C}_{S_0,l_0} \) being given by (28). Moreover, the value function, \( V : \mathcal{C}_{S_0,l_0} \to \mathbb{R}^+ \), of the optimal control problem on an infinite interval \([0, \infty)\) is defined by
\[ V(c_a(t), \tilde{\pi}(t)) = \mathbb{E} \left[ \int_0^\infty \omega (c_a^2(s) + (1 - \omega))l_t(s) - S(s) \right] ds \] (30)

In the sequel, for \( 0 < \omega \leq 1 \), the parameter \( \omega \) is a weighting factor reflecting the relative importance of the capital adequacy and equity risk factors mentioned above. We formulate the value function (30) in the way that we do in order to minimize a combination of the capital adequacy and market risk. This means
that, we would like to minimize

• capital adequacy risk as reflected by the cost on additional bank capital, \( c_a \);
• market risk as reflected by the cost on investment in bank securities, \( \pi \).

We are now in a position to state the stochastic optimal control problem for the risk management of banks in a formal way. The said problem may be formulated as follows.

**Problem 3 (Optimal control problem for bank management).** Assume that \( \mathcal{S}_{S_0,l_0} \neq \emptyset \). Consider the stochastic control system \( (28) \) for the bank risk management problem with the admissible class of control laws, \( \mathcal{S}_{S_0,l_0} \), given by \( (29) \) and the value function, \( V : \mathcal{S}_{S_0,l_0} \rightarrow \mathbb{R}_+ \), given by \( (30) \). Solve

\[
\min_{(c_a(t),\tilde{\pi}(t))\in\mathcal{S}_{S_0,l_0}} V(c_a(t),\tilde{\pi}(t)),
\]

which amounts to determining the optimal value function \( V^* \),

\[
V^*(c_a(t),\tilde{\pi}(t)) = \min_{(c_a(t),\tilde{\pi}(t))\in\mathcal{S}_{S_0,l_0}} V(c_a(t),\tilde{\pi}(t)),
\]

and the optimal control law \( (c_a(t),\tilde{\pi}(t))^* \), if it exists,

\[
(c_a(t),\tilde{\pi}(t))^* = \arg\min_{(c_a(t),\tilde{\pi}(t))\in\mathcal{S}_{S_0,l_0}} V(c_a(t),\tilde{\pi}(t)) \in \mathcal{S}_{S_0,l_0}.
\]

### 3.3. Spread method for bank loan capitalization

In this subsection, we discuss the spread method for loan capitalization that entails the spreading of the unfunded loans over a given spread period. As was mentioned before, the issuing of loans is reliant on the level of macroeconomic activity in the bank’s market. During times of high loan demand, a shortage of capital to finance loans, that have been granted, may arise. One of the ways of dealing with this problem is to adjust the rate of bank capital inflow so that the financing constraint may be removed. In this regard, we may employ a procedure that involves establishing a direct proportionality between the unfunded loans, \( l_u \), and the additional bank capital, \( c_a \). In other words, in order to compensate for an increase (decrease) in loans that are not financed we propose a proportional increase (decrease) in the amount of additional bank capital elicited. A formal definition of the implementability of the spread method for bank loan capitalization is given below.

**Definition 4 (Implementation of spread method for bank loans).** Assume that unfunded bank loans, \( l_u \), are financed by additional bank capital, \( c_a \). The spread method for bank loan capitalization is said to be implementable if there exists a proportionality constant, \( \kappa \), such that

\[
c_a(t) = \kappa l_u(t), \quad t \in [t_0,t_1].
\]

The following lemma provides an existence condition for the spread method for loan capitalization to be implementable. In this regard, we construct the deterministic rate of return on treasuries, \( r \), in such a way that sources of randomness resulting from marketable securities, \( W_i, i = 1,2,\ldots,n \), that are correlated with the Brownian motion, \( W_j \), driving the lending rate, \( l \), are eliminated.

**Lemma 5 (Existence of spread method for loan capitalization).** Suppose that \( \alpha, r, q^T \) and \( \xi \) are characterized by \( (2), (4), (6) \) and \( (7) \), respectively. Then there exists a rate of actualization, \( r_a \) of the form

\[
r_a = r + \sigma^T \tilde{\xi}
\]

that allows a spread method for bank loan capitalization to be implemented.

**Proof.** At the end of the risk period, i.e., \( u = t_1 \), we would like to attain the desired lending rate, \( l(t_1) \). In order to achieve this we can value \( l \) such that

\[
\mathbb{E}(l(t + t_1 - u)|F(t_1)) = \exp\{r(l - r_a)(t_1 - u)\}l(t),
\]

as was done in Theorem 2. We want this valuation to be insensitive to the risk (risk-neutral) associated with stipulated levels of loan issuing. In order for this to be the case, we require that

\[
\mathbb{E} = r - r_a = (r - \alpha^T),
\]

where \( \alpha^T = r - \sigma^T \tilde{\xi} \) enables the growth rate of the lending rate, \( l \), to be amended and its expected value in a risk-neutral market to be the calculated. From \( (32) \) we obtain the expression \( (31) \). \( \Box \)

### 3.4. The main result

We are now in a position to formally state and prove our main result.

**Theorem 6 (Optimal security allocation and capital inflow rate).** Assume that \( (2) \) and \( (31) \) in Lemma 5 hold. Furthermore, suppose that the rate of impatience of the bank, \( r_1 \), is a strict upper bound for \( 2r_1 + \sigma^2 \), i.e.,

\[
2r_1 + \sigma^2 < r_1.
\]

Then the optimal marketable securities allocation may be represented by

\[
\tilde{\pi}(t)^* = \left[ \chi^{-1} \sigma \xi + \sigma_0 q^T \right] (l_u(t) + S(t)) - \chi^{-1} \sigma_0 S(t),
\]

where \( l_u \) is the unfunded loans. Also, the optimal bank capital inflow rate is given by

\[
c(t)^* = c_1(t) + \frac{\delta_{SS}}{\omega} l_u(t),
\]

where \( \delta_{SS} \) is the unique positive solution of the equation

\[
\delta_{SS}^2 + \omega(r_1 - 2r + \tilde{\xi}^T \tilde{\xi}) \delta_{SS} - \omega(1 - \omega) = 0.
\]
Proof. We rely on the procedures suggested in Chapter 5 of Korn (1997) and Chapter 5 of Merton (1990) and the references contained therein. In this regard, we firstly write down the dynamic programming equation (DPE) or Hamilton–Jacobi–Bellman equation (HJBE) by considering the control system (28) and the value function (30). The solution of the DPE is obtained from standard second order partial differential equation theory. In our particular case, we pick a solution that leads to the spread method for bank loan capitalization. Since we have to show that the solution of the control problem and the original value function correspond we complete the proof by verifying that the transversality condition holds (see Chapter 1 of Fleming and Soner, 1993 and p. 49 of Boltyanskii et al., 1962).

For the optimal control problem outlined in Problem 3, the HJBE can be expressed as

\[
\begin{align*}
r_i V(S, l_t) &= \min \{ (r S + \tilde{\pi}^T (\tilde{b} - r \tilde{T}) + c_a \} \\
&\quad + (r_1 - r_a) l_t D_S V(S, l_t) + r_l l_t D_l V(S, l_t) \\
&\quad + \frac{1}{2} \tilde{\pi}^T \tilde{\pi} D_{SS} V(S, l_t) + \frac{1}{2} \sigma_l^2 l_t^2 D_{ll} V(S, l_t) \\
&\quad + \sigma_l \tilde{\pi}^T \sigma_d D_{sl} V(S, l_t) + \omega c_a^2 \\
&\quad + (1 - \omega)(a - l_t)^2. \tag{37}
\end{align*}
\]

Here, \( D_S V(S, l_t), \ D_{SS} V(S, l_t), \ D_{ll} V(S, l_t) \) and \( D_{sl} V(S, l_t) \) denote the first and second partial derivatives of \( V(S, l_t) \) with respect to \( S \) and \( l_t \) in the normal way. Note that the partial differential (37) separates additively into terms

(a) depending on \( c_a \);  
(b) depending on \( \tilde{\pi} \);  
(c) depending on neither \( c_a \) nor \( \tilde{\pi} \).

The minimization problem is therefore decomposed into two separate parts. Minimization (a) is obtained by setting the first derivative of

\[ c_a(D_S V(S, l_t) + \omega c_a) \]

with respect to \( c_a \) equal to zero, i.e.,

\[ D_S V(S, l_t) + 2\omega c_a = 0. \]

It follows that the optimal bank capital inflow rate will be

\[ c_a^*(D_S V(S, l_t)) = - \frac{D_S V(S, l_t)}{2\omega}, \tag{38} \]

where there exists a smooth, strictly convex solution \( V(S, l_t) \) of Eq. (37). Similarly, under these conditions, minimization (b) follows from the optimization of

\[ \tilde{\pi}^T (\tilde{b} - r \tilde{T}) D_S V(S, l_t) + \frac{1}{2} \tilde{\pi}^T \tilde{\pi} D_{SS} V(S, l_t) \]

\[ + \sigma_l \tilde{\pi}^T \sigma_d D_{sl} V(S, l_t) \]

with respect to \( \tilde{\pi} \). In this case, the optimal security allocation strategy may be written as

\[
\begin{align*}
\tilde{\pi}^* (D_S V(S, l_t), D_{SS} V(S, l_t), D_{sl} V(S, l_t)) &= - \chi^{-1} (\tilde{b} - r \tilde{T}) \frac{D_S V(S, l_t)}{D_{SS} V(S, l_t)} \\\n&\quad - \sigma_l \sigma^{-1} q \frac{D_{sl} V(S, l_t)}{D_{SS} V(S, l_t)}. \tag{39}
\end{align*}
\]

Substituting the optimal values \( c_a^* \) and \( \tilde{\pi}^* \) given by (38) and (39), respectively, into the right-hand side of (37) yields

\[
\begin{align*}
\{ r S - (\tilde{b} - r \tilde{T}) \chi^{-1} (\tilde{b} - r \tilde{T}) \frac{D_S V(S, l_t)}{D_{SS} V(S, l_t)} \}
&\quad - \frac{D_S V(S, l_t)}{2\omega} + (r_1 - r_a) l_t \} \frac{D_S V(S, l_t) + r_l l_t D_l V(S, l_t) + \omega \sigma_l^2 l_t^2}{D_{SS} V(S, l_t)} \\
&\quad + \frac{1}{2} \left\{ (\tilde{b} - r \tilde{T}) \chi^{-1} (\tilde{b} - r \tilde{T}) \right\} \frac{D_S V(S, l_t)}{D_{SS} V(S, l_t)} \\
&\quad + \frac{\tilde{b} - r \tilde{T}}{D_{SS} V(S, l_t)} \chi^{-1} \frac{D_{sl} V(S, l_t)}{D_{SS} V(S, l_t)} \\
&\quad + \tilde{\pi}^T \tilde{\pi} D_{ll} V(S, l_t) \frac{D_S V(S, l_t) + \omega \sigma_l^2 l_t^2}{4\omega} \\
&\quad + (1 - \omega)(S - 2S l_t + l_t^2). \tag{40}
\end{align*}
\]

By standard second order partial differential equation theory, the form of (40) suggests that

\[ V(S, l_t) = \delta_{SS} S^2 + \delta_{ll} l_t^2 + \delta_{sl} S l_t \]

is a quadratic solution of (37). Here \( \delta_{SS} \), \( \delta_{ll} \), and \( \delta_{sl} \), are solutions of

\[
\begin{align*}
\delta_{SS}^2 + \omega (r_1 - \tilde{\xi} \tilde{\tau}) - 2\omega(1 - \omega) \delta_{sl} - \omega (1 - \omega) &= 0, \tag{42} \\
4\omega (r_1 - \tilde{\xi} \tilde{\tau}) - \sigma_l^2 l_t^2 \delta_{ll} + \delta_{SS} \delta_{sl} - 4\omega (r_1 - r_a) \delta_{sl} \delta_{SS} + \omega (\sigma_l^2 \tilde{\xi} \tilde{\tau} - 2\omega(1 - \omega)\delta_{SS} &= 0, \tag{43}
\end{align*}
\]
This approach to calculating the optimal bank capital inflow\(^7\) bank loan capitalization we proceed in a more specific manner. of the parameters of the problem that lead to a spread method for bank loan capitalization we proceed in a more specific manner.

We note that (42)–(44) can be solved in a general way using standard theory. However, since we seek a solution with values above is that the non-negativity of the value function in (41) is a homogeneous geometric Brownian motion with the result that

\[
\lim_{t \to \infty} \exp(-r_t t) h(t) = 0
\]

if and only if (33) holds. From Chapter 8 of \textit{Arnold (1974)}, \(f\) and \(g\) in (48) satisfy the linear differential equations

\[
f'(t) = \left( r - \frac{\gamma T \xi}{\omega} - \frac{\delta SS}{\omega} + r_i - \sigma q_l \right) f(t)
+ \left( -r + \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} + r_i - \sigma q_l \right) h(t),
\]

\[
g'(t) = 2 \left( r - \frac{\gamma T \xi}{\omega} - \frac{\delta SS}{\omega} \right) g(t)
+ 2 \left( -r + \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} + r_i - \sigma q_l \right) f(t),
\]

\[g(0) = S_0^*,\]

respectively. As a consequence, we are able to conclude that

\[f(t) = (S_0 - k_1 l_0) l_0 \exp \left\{ \left( r - \frac{\gamma T \xi}{\omega} - \frac{\delta SS}{\omega} + r_i - \sigma q_l \right) t \right\} + k_1 h(t),\]

where the constant

\[k_1 = \frac{\delta SS + \sigma q \left( r + \frac{\gamma T \xi}{\omega} + r_i - \sigma q_l \right)}{\delta SS + \sigma q \left( r + \frac{\gamma T \xi}{\omega} + r_i - \sigma q_l \right)}.
\]

It immediately follows that

\[
\lim_{t \to \infty} \exp(-r_t t) f(t) = 0
\]

if and only if inequalities (33) and

\[r - \frac{\gamma T \xi}{\omega} - \frac{\delta SS}{\omega} + r_i - \sigma q_l \frac{\gamma T \xi}{\omega} < r_i\]

both hold simultaneously. Next, we conjecture that the rate of impatience inequality given by (33) and the definition of \(\delta SS\) as the positive solution of (36) give rise to (49). This conclusion is reached by observing for \(-1 \leq q_l \leq 1\)

\[a_l \frac{\gamma T \xi}{\omega} - \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} \leq \frac{1}{2} \left( \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} \right) \left( \frac{\gamma T \xi}{\omega} - \frac{\gamma T \xi}{\omega} \right) \leq \frac{1}{2} q_l^2.
\]

Also, we have that (36) implies that

\[r - \frac{\gamma T \xi}{2} - \frac{\delta SS}{\omega} < r_i\]

We recall from an earlier discussion that \(l_i\) is modelled as geometric Brownian motion with the result that

\[h(t) = l_i^2 \exp((2r + \sigma_l^2) t)\] and

\[
\lim_{t \to \infty} \exp(-r_t t) h(t) = 0
\]

It immediately follows that

\[
\lim_{t \to \infty} \exp(-r_t t) f(t) = 0
\]

if and only if inequalities (33) and

\[r - \frac{\gamma T \xi}{\omega} - \frac{\delta SS}{\omega} + r_i - \sigma q_l \frac{\gamma T \xi}{\omega} < r_i\]

both hold simultaneously. Next, we conjecture that the rate of impatience inequality given by (33) and the definition of \(\delta SS\) as the positive solution of (36) give rise to (49). This conclusion is reached by observing for \(-1 \leq q_l \leq 1\)

\[a_l \frac{\gamma T \xi}{\omega} - \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} \leq \frac{1}{2} \left( \frac{\gamma T \xi}{\omega} + \frac{\delta SS}{\omega} \right) \left( \frac{\gamma T \xi}{\omega} - \frac{\gamma T \xi}{\omega} \right) \leq \frac{1}{2} q_l^2.
\]

Also, we have that (36) implies that

\[r - \frac{\gamma T \xi}{2} - \frac{\delta SS}{\omega} < r_i\]
4.2. Optimal risk management of banks

In this subsection, we consider optimal security allocation and capital inflow strategies suggested by our main results.

4.2.1. Optimal security allocation

The first observation that we make is that the optimal security allocation given by (34) in Theorem 6 may be re-written as

$$\tilde{\pi}^a = \beta^{-1}(\bar{b} - r \bar{t})l_u + \sigma_l \sigma^T \tilde{q} t. \quad (52)$$

The form of this optimal solution to our control problem intimates that $l_u$ and $\tilde{\pi}^a$ are directly related to each other with a constant of proportionality (also known as the optimal growth portfolio strategy) depending on $\sigma$ and the constants $\beta$ and $r T$. Of course, we may also introduce the market price of risk to this proportionality via (7). The term $\sigma_l \sigma^T \tilde{q} t$ in (52) corrects for risk and suggests that our optimal security allocation strategy is dependent on the uncertainty in the loan issuing and its reference process, the risk constraints in the model and the correlation between loans and returns on securities.

Next, we discuss three optimal strategies for security allocation that follow from Theorem 6, viz., those involving borrowing, short selling and overcapitalization. In order to facilitate a discussion of the optimal security allocation strategy, we set

$$M_i = \tilde{\xi}_i \beta^{-1} \sigma_l^r \bar{q} \ \text{and} \ \frac{N_i}{M_i} = \alpha_i \tilde{e}_i \sigma^{-T} \tilde{q} t. \quad (50)$$


4.4. Discussion of results

In this section, we comment on the main issues raised in the discussion above. In particular, we consider the main features of the stochastic model that we construct and optimal procedures suggested by Theorem 6.

4.4.1. Stochastic model of a bank

Our continuous-time stochastic model is based on the stylized balance sheet presented in (1). A choice of banking policy which aims to correlate bank borrowings and portfolio features ultimately leads to the SDE (28).

Many aspects of our model are consistent with US commercial bank data from the last 20 years as reported by the Federal Deposit Insurance Corporation (FDIC) (2005). Also, our model has similarities with the one proposed in Thakor (1996) in which bank capital and loan issuing is aligned. Further connections with Dangl and Lehar (2004); Decamps et al. (2004) and Diamond and Rajan (2000) where continuous-time models are constructed for the dynamics of portfolio and capital structure management, appear to be evident. Moreover, the stochastic systems that we obtain are related to the work on bank portfolio choice presented in Hellmann, Murdock, and Stiglitz (2000).
The aforementioned control problem is solved by imposing a
process, \( c(t) \), and the BC reference function, \( r_0 \), have for \( e^c \) given by (35) that

\[
E_{t_0}c(t) - E_{t_0}c(t)
\]

exhibits the same behaviour as time approaches infinity as is the case for security allocation in Section 4.2.1.

4.2.2. Optimal capital inflow
For the bank capital inflow rate, \( c \), and the BC reference process, \( c(t) \), we have for \( e^c \) given by (35) that

\[
E_{t_0}c(t) - E_{t_0}c(t)
\]

exhibits the same behaviour as time approaches infinity as is the case for security allocation in Section 4.2.1.

5. Conclusion
The main novelty of our paper is the solution of an optimal stochastic control problem that minimizes bank market and capital adequacy risks by making choices about security allocation and capital requirements, respectively. The former is measured by the deviations of the bank’s securities from the loan issuing (LI) process and is an indicator of the bank’s safety. The latter provides information about the size of the deviation of bank capital requirements from the bank capital (BC) reference process and is related to the financial stability of the bank. The aforementioned control problem is solved by imposing a spread method for bank loan capitalization. In reality, the vast majority of commercial banks in the USA are well-capitalized. From the FDIC banking profile for the fourth quarter of 2005, the situation was that more than 99.8% of industry assets, met or exceeded the highest regulatory capital requirements. Roleplayers in the banking industry agree that capital adequacy must be discussed in terms of the regulation of all facets of banking activity. It is thus imperative that future optimal control problems should be formulated in such a way that they incorporate the three pillars of the Basel II capital accord, viz., minimum regulatory capital requirement (Pillar 1); supervisory review (Pillar 2) and market discipline (Pillar 3). This will require the construction of new and improved models for bank behaviour. Furthermore, the new banking accord has heightened interest in risk management and accompanying regulatory policy. In this regard, we can identify three areas of particular importance. Firstly, unpacking the relationship between different types of risk and their management has become a research priority. In particular, issues associated with empirical and theoretical linkages between credit and market risk and approaches to the integrated measurement and management of credit, market, operational and liquidity risk (see, for instance, Basel II, September 2001; Fouche, Mukuddem-Petersen, and Petersen, 2006; Mukuddem-Petersen and Petersen, 2005; Petersen et al., 2005 for further discussion) are of considerable interest. Secondly, developing a better understanding of the ramifications of linkages between different types of risks has become an imperative. In this case, the scope for integrated banking regulation between different types of risk, theoretical and empirical modelling of economic versus regulatory capital and the implications of and interactions between the three pillars of Basel II for risk management (including pro-cyclicality) requires further investigation. Finally, incentives, risk transfers and systemic risk are interesting ongoing topics. More specifically, this involves agency problems in financial institutions and incentives for risk management, risk transfers within banks (e.g., between banking and trading book) and among financial institutions and implications of incentives and risk transfers for the nature of systemic risk in the financial system and for the structure of the financial industry.

Acknowledgements
We would like to thank the associate editor, the special issue guest editor and three anonymous referees for their insightful comments and suggestions. Furthermore, we acknowledge the financial support by the National Research Foundation (NRF) of South Africa under GUN Numbers 2053343 and 2074218.

References


Janine Mukuddem-Petersen was born in Matroosfontein, Cape Town and received her Ph.D. degree from the North-West University, Potchefstroom (NWU-PC), South Africa, in 2005. Her main research interests are in economic systems theory and health science. She is currently a research fellow in financial and health economics at the NWU-PC, having completed a postdoctoral tenure at the Vrije Universiteit, Amsterdam, the Netherlands in 2005. Mukuddem-Petersen has won several awards and fellowships for her ongoing research into the economic evaluation of financial and health systems.

Mark Petersen was born in Elsies River, Cape Town and received his Ph.D. degree in Mathematics from the University of the Western Cape, Bellville, South Africa, in 1998. His research interests include operator theory, matrix analysis, nonlinear control systems and the mathematical theory of economic systems. Petersen is the author of more than 40 peer-reviewed publications in these areas. He is currently a professor of mathematics at the North-West University in Potchefstroom, South Africa. Short term visits elsewhere include Virginia Tech, Blacksburg, USA, Tilburg Universiteit and the Vrije Universiteit and Centrum voor Wiskunde en Informatica, Amsterdam, the Netherlands. Petersen has been appointed to several international technical committees and working groups in the research areas of applied and theoretical mathematics, modelling, simulation and economics.