Optimal auditing in the banking industry

T. Bosch¹, J. Mukuddem-Petersen², M. A. Petersen², *, † and I. Schoeman²

¹Underwriting Department, Santam Insurers, P.O. Box 3881, Tyger Valley, Cape Town 800, South Africa
²Department of Mathematics and Applied Mathematics, North-West University, Private Bag x6001, Potchefstroom 2520, South Africa

SUMMARY

As a result of the new regulatory prescripts for banks, known as the Basel II Capital Accord, there has been a heightened interest in the auditing process. Our paper considers this issue with a particular emphasis on the auditing of reserves, assets and capital in both a random and non-random framework. The analysis relies on the stochastic dynamic modeling of banking items such as loans, reserves, Treasuries, outstanding debts, bank capital and government subsidies. In this regard, one of the main novelties of our contribution is the establishment of optimal bank reserves and a rate of depository consumption that is of importance during an (random) audit of the reserve requirements. Here the specific choice of a power utility function is made in order to obtain an analytic solution in a Lévy process setting. Furthermore, we provide explicit formulas for the shareholder default and regulator closure rules, for the case of a Poisson-distributed random audit. A property of these rules is that they define the standard for minimum capital adequacy in an implicit way. In addition, we solve an optimal auditing time problem for the Basel II capital adequacy requirement by making use of Lévy process-based models. This result provides information about the optimal timing of an internal audit when the ambient value of the capital adequacy ratio is taken into account and the bank is able to choose the time at which the audit takes place. Finally, we discuss some of the economic issues arising from the analysis of the stochastic dynamic models of banking items and the optimization procedure related to the auditing process. Copyright © 2007 John Wiley & Sons, Ltd.

Received 22 August 2006; Revised 18 June 2007; Accepted 27 August 2007

KEY WORDS: stochastic modeling and optimization; bank auditing
1. INTRODUCTION

Most commercial and investment banks are subjected to regular internal and/or external (for example, by regulators) auditing of operational items such as assets, liabilities and capital. An audit involves the evaluation of the adequacy of the bank’s systems of internal control, the extent of compliance with established procedures and regulations and the effectiveness of banking operations. Despite the importance of this process, it has not been discussed very extensively in the banking literature when compared with other financial decision-making problems such as asset quality and credit risk assessment. We separately discuss scenarios in which audits may take place on a random (particularly in the case of external audits) and non-random (especially for internal audits) basis (see, for instance, [1–6]). Importantly, we note that the latter option invariably allows the bank owner to exert some influence over the timing of the audit. Of course, the vast majority of audits undertaken in the banking industry are random and involve external auditing. Usually, the conclusions drawn from internal audits are made available to external auditors and as a result has some effect on the outcome of the latter type of audit.

Three operational requirements that play an important role in auditing are related to the reserves and capital held by the bank and the setting of rules for its default and closure. With regard to the former, it is important for auditors to be able to measure the volume of (Treasuries and) reserves that the bank holds in order to cater for anticipated and unanticipated deposit withdrawals. This auditing criterion is known as the reserve requirement and considers the nature of the reserves that are available to meet the bank’s obligations to depositors. Both auditors and banks are interested in establishing the optimal level of reserves on demand deposits that the bank must hold. By setting a bank’s individual optimal level of reserves, auditors assist in mitigating the costs of financial distress. For instance, if the minimum level of required reserves exceeds a bank’s optimally determined level of reserves, this may lead to deadweight losses. Secondly, we explore the connection between auditing and asset requirements that are formulated by the bank’s shareholders and regulators. In this regard, in the case of a Poisson-distributed random audit, the unlevered asset value, $A$, is allowed to evolve without any restriction on time until it becomes less than a critical asset value, $A^r$, that is chosen by the shareholders and initiates the default process. In addition, we discuss a related prescribed asset value, $A^r$, set by the regulator, which is instrumental in determining a threshold for bank closure and reorganization. The problem of determining and characterizing $A^r$ and $A^r$ and their interrelationship is sometimes called the asset threshold problem. Finally, auditors are interested in the capital requirement that aims to determine the sufficiency of the capital held by the bank for reducing the default risks on deposits, as well as in incentivising sensible risk-taking. For these issues, empirical and theoretical linkages between credit and market risk and approaches to the measurement and management of credit, market, liquidity and operational (see, for instance, [7]) risk are of considerable interest. In addition, the scope for integrated banking regulation between different types of risk and theoretical and empirical managements of economic versus regulatory capital impacts management techniques.

Banks are among the most heavily regulated of all financial institutions. In particular, reserve and capital requirements have become important components of regulation and supervision in the banking industry. As from June 1999, the Basel Committee on Banking Supervision (BCBS) released several proposals (see, for instance, [8–10]) to reform the original 1988 Basel Capital Accord (see [11]). These efforts culminated in the Basel II Capital Accord (see, for instance, [12–15]) which is based on three pillars (see [16] for a discussion on the interaction between these pillars). Pillar 1 intends to provide a stronger link between the management of capital
requirements and actual risk. Pillar 2 focuses on strengthening the supervisory process, particularly in assessing the quality of risk management in banking institutions and in evaluating whether these institutions have adequate procedures to determine how much capital they need. Pillar 3 involves the improvement of market discipline through increased disclosure of details about the bank’s credit exposures, its amount of reserves and capital, the bank owners and the effectiveness of its internal ratings system. Since bank management has become increasingly complicated and supervisors (acting as representatives of the depositors’ interests) battle to monitor banking activities, the recourse to market discipline appears to be justified. In this regard, monitoring of banks by professional investors and financial analysts as a complement to banking supervision is being encouraged. However, the manner in which market discipline and the other two pillars are to be managed in concert with each other is a subject of much debate.

In Basel II, the ratio of capital to assets, also called the capital adequacy ratio (CAR) plays a major role as an index of the sufficiency of capital held by banks. This ratio is the centerpiece of the minimum capital requirement (Pillar 1) and has the form

\[
\text{Capital adequacy ratio} = \frac{\text{Indicator of absolute amount of capital}}{\text{Indicator of absolute level of risk}}
\]

Our study expresses the CAR as

\[\text{Basel CAR} (\rho) = \frac{\text{Value of bank capital (C)}}{\text{Value of risk-weighted assets (a)}}\] (1)

In practice, the CAR value, \(\rho\), is allowed to evolve without any constraints on time until it becomes less than a certain CAR value, \(\rho^s\), that initiates the default process. Usually \(\rho^s\) is chosen by the bank’s shareholders who generally regard capital adequacy requirements as being indicative of the financial health of the bank. An additional scenario is that an audit may occur and the regulator may decide to close the bank because \(\rho\) is below a prescribed value \(\rho^r\), which is set by the regulator. We note that both the capital thresholds, \(\rho^s\) and \(\rho^r\), are exogenously chosen and that invariably \(\rho^s \leq \rho^r\). Despite the fact that Basel II gives a precise description of the refinements to the RWAs to be used in the computation of \(\rho\), it neglects to describe \(\rho^r\). In this regard, Figure 1 (see [17] for more details) provides information about how the Basel CAR from (1) and the leverage CAR

<table>
<thead>
<tr>
<th>CATEGORIES</th>
<th>(\rho)</th>
<th>T1CAR</th>
<th>(\rho^L)</th>
<th>TE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well-Capitalized</td>
<td>(\geq 0.1) and (\geq 0.06) and (\geq 0.06)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adequately Capitalized</td>
<td>(\geq 0.08) and (\geq 0.04) and (\geq 0.04)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Undercapitalized</td>
<td>(\geq 0.06) and (\geq 0.03) and (\geq 0.03)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Significantly Undercapitalized</td>
<td>(&lt; 0.06) or (\geq 0.03) or (\geq 0.03) and (&gt; 0.02)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critically Undercapitalized</td>
<td></td>
<td></td>
<td>(\leq 0.02)</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1. Categories of banking threshold regulatory ratios.
Leverage CAR \( (\rho^L) \) may be categorized in terms of supervisory risk groupings. In Figure 1, we have that T1CAR and TE are the abbreviations for the Tier 1 CAR and tangible equity, respectively.

1.1. Preliminaries

In the sequel, we use the notational convention ‘subscript \( t \) or \( s \)’ to represent (possibly) random processes, while ‘bracket \( t \) or \( s \)’ is used to denote deterministic processes. Our contribution concentrates on the dynamics of banking items with jump diffusions. Such processes have an advantage over the more traditional modeling tools such as Brownian motion in that they allow for the non-continuous evolution of the value of such items. In order to formalize this notion, we suppose for \( F = (\mathcal{F}_t)_{t \geq 0} \) that \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a filtered probability space. Also, we recall that an \( \mathcal{F}_t \)-adapted process \( \{L_t\}_{t \geq 0} \subset \mathbb{R} \) with \( L_0 = 0 \) a.s. is known as a Lévy process if \( L_t \) is continuous in probability and has stationary, independent increments. \( L_t \) has a cadlag version (right continuous with left-hand limits) which is also a Lévy process. We will assume that the type of such processes that we work with are always cadlag. The jump of \( L_t \) at \( t \geq 0 \) is defined by \( \Delta L_t = L_t - L_{t^-} \). Furthermore, we let \( \mathcal{B}_0 \) be the family of Borel sets \( \mathcal{U} \subset \mathbb{R} \) with \( 0 \notin \mathcal{U} \). In addition, for \( \mathcal{U} \subset \mathcal{B}_0 \), we define the Poisson random or jump measure of \( L \) as

\[
N(t, \mathcal{U}) = N(t, \mathcal{U}, \omega) = \sum_{0 < s \leq t} \mathcal{X}_\mathcal{U}(\Delta L_s)
\]

This means that \( N(t, \mathcal{U}) \) is the number of jumps of size \( \Delta L_s \in \mathcal{U} \) which occurs before or at time \( t \). The differential form of \( N(t, \mathcal{U}) \) is \( N(dt, dx) \). The set function

\[
v(\mathcal{U}) = \mathbb{E}[N(1, \mathcal{U})]
\]

where \( \mathbb{E} = \mathbb{E}^\mathbb{P} \) denotes expectation with respect to \( \mathbb{P} \), defines a \( \sigma \)-finite measure on \( \mathcal{B}_0 \) called the Lévy measure of \( \{L_t\}_{t \geq 0} \). Finally, the compensated Poisson random measure of \( \{L_t\}_{t \geq 0} \) is given by

\[
\tilde{N}(dt, dx) = N(dt, dx) - v(dx) \, dt
\]

Subsequently, if \( v = 0 \) then we will have that \( L_t = Z_t \), where \( Z_t \) is appropriately defined Brownian motion.

By considering the discussion above, we may assume that the bank’s unlevered asset value, \( A \), has Lévy process-driven dynamics of the form

\[
dA_t = A_t \{[\mu(A_t, t) - f_t] \, dt + \sigma_t \, dZ_t\} + A_t \int_{-1}^\infty \xi \tilde{N}(dt, dx)
\]

where \( \mu(A_t, t) \) is the total expected rate of return on \( A \), \( f_t \) is the fraction of \( A \) paid to shareholders, i.e. net cash outflow, \( \sigma_t \) is the proportional volatility of \( A \) per unit time and \( dZ \) is the increment of a standard Brownian motion.
In the case of the random audit process, $A$, the regulator’s audits of the bank’s asset value, $A$, may be assumed to be stochastic and follow a Poisson process, where the mean number of audits per time unit is denoted by $m$. This may be expressed in a formal way via the stochastic process

$$dA = dv, \quad A = \begin{cases} 1 & \text{audit occurs} \\ 0 & \text{otherwise} \end{cases}$$

Note that under the above assumptions the probability that a random audit occurs in the time interval $dt$ is $m dt$. On the other hand, the probability of no audit is $1 - m dt$ and the probability of more than one random audit is of the order $o(dt)$. Furthermore, we suppose that the two stochastic processes $dZ$ and $dv$ are independent. This description of the random audit has particular relevance for the analysis of the bank default and closure thresholds, $A^d$ and $A^c$, respectively. This procedure is via unlevered assets that are exogenously set by the shareholders and regulators. In our contribution, it is of lesser importance for the random auditing of reserve requirements.

In the discussion on capital requirements, we cater for the possibility that the bank is able to choose the time at which a non-random audit takes place. Although this seems to violate the principles on which the (external) auditing process is based, we make this assumption to cater for the situation that may arise for internal audits. It may be that a worst stopping time analysis is more appropriate in this case. However, that approach is beyond the scope of this paper but would make for an interesting topic for future research.

1.2. Relationship to previous literature

Our work extends aspects of the paper [18] by generalizing the description of bank behavior in a continuous time Brownian motion framework to one in which the dynamics of bank items may have jumps and be driven by Lévy processes. As far as information on these processes is concerned, Protter [19, Chapter I, Section 4] or Jacod and Shiryaev [20, Chapter II] are standard texts. Also, the connections between Lévy processes and finance are embellished upon in [21]. As far as banking regulation is concerned, we note that formal models that enable aspects of Basel II to be analyzed is suggested in [16, 22]. In fact, when discussing the main issues that arise from the models in our paper, we make liberal use of the investigations undertaken in these contributions (see, also [23–25]).

The most important role of capital is to mitigate the moral hazard problem that results from asymmetric information between banks, depositors and borrowers. The Modigliani–Miller theorem forms the basis for modern thinking on capital structure (see [26]). In an efficient market, their basic result states that, in the absence of taxes, insolvency costs and asymmetric information, the bank value is unaffected by how it is financed. In this framework, it does not matter if bank capital is raised by issuing equity or selling debt or what the dividend policy is. By contrast, in our contribution, in the presence of loan market frictions, the value of the bank is dependent on its financial structure (see, for instance, [4, 27–29] for banking). In this case, it is well known that the bank’s decisions about lending and other important issues may be driven by the CAR (see, for instance [16, 22, 30–32]). Further evidence of the impact of capital requirements on the lending activities of banks are provided by [33, 34]. Also, our model allows for the fact that the banking industry may not be perfectly competitive either because of collusion, interest ceilings on deposits or government subsidies that provide a tax-shield on interest.

As far as the literature on auditing in the banking industry is concerned, we highlight the contributions made in [1–6]. In the former paper, a model of optimal bank default and closure

rules with Poisson-distributed audits of the bank’s asset value by the regulator, with the goal of eliminating the incentives of the levered bank shareholders/managers to take excessive risks in their choice of the underlying assets, were studied. The roles of (tax or other) subsidies on deposit interest payments by the bank and of the auditing frequency are examined. The paper [2] examines alternative ways to prevent losses from bank insolvencies. In particular, this contribution develops a model that compares two alternative institutions for bank auditing. The first is a system of central bank auditing of national banks. The second is carried out by an international agency that collects and disseminates risk information on banks in all countries. The contribution [3] suggests that auditing systems can be effective devices to counteract tendencies for risk-taking associated with bank safety nets. Results are obtained from an international sample of publicly traded banks after controlling for other regulatory control devices for bank risk, such as restrictions on banking activities, minimum regulatory capital requirements and official discipline. The efficacy of auditing systems in controlling bank risk diminishes with bank charter value and increases with moral hazard stemming from a country’s deposit insurance. The results also indicate that auditing systems are complements for minimum capital requirements, but substitutes for restrictions on bank activities and official discipline. Furthermore, we consider our discussion on audit times and thresholds to be related to [5]. In the contribution [6], the objective is to explore the potentials of developing multicriteria decision-aid models for reproducing, as accurately as possible, the auditors opinion on the financial statements of the firms.

1.3. Outline of the paper

Under the conditions highlighted above, the main problems addressed in our contribution can be identified as follows. In Section 2, we extend [35] (see, also, [31, 36]) by presenting jump diffusion models for the unlevered assets, $A$, loan supply and demand, $A$, reserves, $R$, bank capital, $C$, and CARs, $\rho$. In turn, the model for the loan demand takes changing macroeconomic activity, $M$, into account. We also bear in mind that deposit withdrawals are catered for by the Treasuries and reserves held by the bank. The stochastic dynamics of both these items and their sum are presented in Section 2.1. This enables us to express the dynamics of the total unlevered assets in terms of the loans, Treasuries and reserves in Sections 2.1.3 and 2.3. Our discussion on bank debt is less general and takes place within a Brownian framework. Here, we are able to deduce an explicit formula for the outstanding debt in terms of a coupon rate, the Treasuries rate, the unlevered assets and the shareholder default and regulator closure rules, $A^\delta$ and $A^\gamma$, respectively. In a manner similar to that for debts, we solve an ordinary differential equation (ODE) to find a formula for the subsidy. Also, additional items such as the net cash outflow, $f$, and the auditing rate, $m$, are introduced in a dynamic modeling framework.

Section 3 contains the analysis of the auditing process with emphasis on reserve, asset and capital requirements. In this regard, one of the main novelties of our paper is encapsulated by Theorem 3.2 in Section 3.1, where the optimal proportion of bank reserves and rate of depository consumption are determined via Lévy processes. In this case, the rate of depository consumption is defined as the rate at which Treasuries and reserves are consumed by the taking, holding and anticipated withdrawal of deposits. Typically, the bank owner has to make decisions about deposit taking via the fixing of costs related to cheque clearing and bookkeeping, the holding of deposits by means of the choice of the deposit and coupon rate and anticipated withdrawals via the provisioning provided by Treasuries and reserves. Examples of anticipated withdrawals are stop orders, anticipated living expenses and certain payments. Also, the specific choice of a power utility function for the bank
OPTIMAL AUDITING IN THE BANKING INDUSTRY

owner is made in order to obtain an analytic solution. Section 3.2 provides explicit formulas for the shareholder default and regulator closure rules, $A^*$ and $A^r$, respectively. The main result in this subsection, viz., Theorem 3.3, has a heavy reliance on the analysis of random auditing contained in [1] (see, also, [2, 4, 5]). Theorem 3.5 in Section 3.3 solves an optimal (non-random) auditing problem in terms of the CAR in a Lévy process setting. This result provides information about the optimal timing of an audit by the regulator when the ambient value of the CAR is taken into account and the bank is able to choose the time at which the audit takes place.

In Section 4, we analyze the main economic issues arising from the stochastic banking model and the auditing process. Finally, Section 5 offers a few concluding remarks. The appendices serve several purposes. In the main, it provides a theoretical basis and/or outline of the proofs of Theorems 3.2, 3.3 and 3.5.

2. STOCHASTIC MODEL FOR BANKS

The Basel II capital accord allows us to construct a continuous-time stochastic dynamic model that consists of assets, $A$, (uses of funds), liabilities, $\Gamma$, (sources of funds) and bank capital, $C_t$, (see, for instance [28]). In our contribution, these items can specifically be identified as

$$A_t = \Lambda_t + T_t + R_t, \quad \Gamma_t = \Delta_t, \quad C_t = n_t E_t + O_t$$

where $\Lambda, T, R, \Delta, n, E$ and $O$ are the current value of the loans, Treasuries, reserves, outstanding debt, number of shares, bank equity and subordinate debt, respectively. Our model allows for the banking industry to be imperfectly competitive because of, for instance, government subsidies. The latter mentioned item we denote by $S$ with the value of capital, $C_t$, being the total value of the bank, $A_t + S_t$, minus the value of its debt, $\Delta_t$, as given by

$$n_t E_t + O_t = A_t + S_t - \Delta_t \quad (4)$$

Banks may also receive a safety net subsidy that lowers their cost of capital.

2.1. Assets

In this subsection, the bank assets that we discuss are loans, Treasuries, reserves, total unlevered assets and risk-weighted assets. In the sequel, we suppose that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space.

2.1.1. Loans. We suppose that, after providing liquidity, the bank grants loans at the interest rate on loans or loan rate, $r^L_t$. Profit maximizing banks set their loan rates, $r^L_t$, as a sum of the risk-free Treasuries rate, $r^T$, the expected loss ratio, $E(d)$, and of the risk premium, $k$. Here, the risk premium, $k$, under the CAPM model could be quantified by the relation

$$k = \beta (r^m - r^T)$$

where $r^m$ is the rate of return of the market portfolio. Furthermore, expressing the expected losses, $E(d)$, as a rate of return per unit time, we obtain the expression

$$r^L_t = r^T + k + E(d)$$

The sum $r^T + k$ provides the remuneration for the cost of monitoring and screening of loans and of capital, $c^A$. The $E(d)$ component is the amount of provisioning that is needed to match the average losses faced by the loans. The representation of the banks’ interest setting shows that banks will experience excess returns in good times when the actual rate of default, $r^d$, is lower than the provisioning for expected losses, $E(d)$, and will not be able to cover their expected losses when $r^d > E(d)$. In this case, bank capital will be needed to cover these excess losses. If this capital is not enough then the bank will face insolvency.

In this paragraph, we provide a brief discussion of loan demand and supply. Firstly, we introduce the generic variable, $M_t$, that represents the level of macroeconomic activity in the bank’s loan market. We suppose that macroeconomic process, $M = \{M_t\}_{t \geq 0}$, follows the Lévy process

$$dM_t = M_t \{\mu_t^M dt + \sigma_t^M dZ_t^M\} + M_t - \int_{-1}^{\infty} x^M \tilde{N}^M (dt, dx^M)$$

where $\sigma_t^M$ and $Z_t^M$ denote volatility in macroeconomic activity and the Brownian motion driving the macroeconomic activity, respectively. In fact, in the sequel, the actual default rate, $r^d$, can be considered to be dependent on macroeconomic conditions, $M$, and be denoted by $r^d(M_t)$.

Taking our lead from the equilibrium arguments in [37], we denote both the supply and demand credit price processes by $A = \{A_t\}_{t \geq 0}$. In this regard, the bank faces a *Hicksian demand for loans* given by

$$\Lambda_t = \lambda_0 - l_1 \int_0^t \Lambda_s \, ds + \int_0^t \sigma_s^A \, dZ_s^A + l_2 M_t$$

(5)

where $\sigma_t^A$ and $Z_t^A$ denote volatility in the loan demand and the Brownian motion driving the demand for loans (which may be correlated with the macroeconomic activity), respectively. We note that the loan demand in (5) is an increasing function of $M$ and a decreasing (increasing) function of $\int_0^t \Lambda_s \, ds > 0 (< 0)$. Also, we assume that the *price process of the loan supply*, $\Lambda = \{\Lambda_t\}_{t \geq 0}$, follows the geometric Lévy process

$$d\Lambda_t = \Lambda_t \{\mu_t^A dt + \sigma_t^A dZ_t^A\} + \Lambda_t - \int_{-1}^{\infty} x^A \tilde{N}^A (dt, dx^A) - f_t A_t \, dt$$

(6)

where $f_t$ is the fraction of $A$ paid to shareholders, $\mu_t^A = r_t^A - c^A - r^d(M_t)$, $\sigma_t > 0$ denotes the volatility in the loan supply and $Z_t$ is a standard Brownian motion with respect to a filtration, $\mathcal{F}_t$, of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. The *value of the bank’s investment in loans*, $\lambda$, at $t$ is expressed as

$$\lambda_t = n_t^A \Lambda_t$$

(7)

where $n_t^A$ is the number of loans at $t$. For the sake of argument, in the sequel, we assume that $n_t^A = n^A = 1$ in (7) so that $\lambda_t = \Lambda_t$, for all $t$.

### 2.1.2. Treasuries and reserves

**Treasuries and reserves.** Treasuries, $T$, are bonds issued by national Treasuries. They are the debt financing instruments of the federal government, and are often referred to as ‘Treasuries.’ There are four types of Treasuries, viz., Treasury bills, Treasury notes, Treasury bonds and savings bonds. All of the Treasuries besides savings bonds are very liquid and are heavily

---


---
traded on the secondary market. We denote the interest rate on Treasuries or Treasury rate by $r^T_t$ and assume that for all $t$ we have

$$
\mu_t^A = r^A_t - c^A - r^d(M_t) > r^T_t
$$

Bank reserves are the deposits held in accounts with a national agency (for instance, the Federal Reserve for banks) plus money that is physically held by banks (vault cash). Such reserves are constituted by money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits will be needed as reserves. As a result of this description, we may introduce a reserve–deposit ratio, $\gamma$, for which

$$
R_t = \gamma \Delta_t
$$

The bank uses the remaining deposits to earn profit, either by issuing loans or by investing in assets such as Treasuries and stocks. The individual rationality constraint implies that reserves may implicitly earn at least their opportunity cost through certain bank operations and Federal government subsidies. For instance, members of the Federal Reserve in the United States may earn a return on required reserves through government debt trading, foreign exchange trading, other Federal Reserve payment systems and affinity relationships (outsourcing) between large and small banks. We note that vault cash in the Automated Teller Machines (ATMs) network also qualifies as required reserves. Also, the investment of bank reserves in the market via bonds and stocks is still possible in many countries. The conclusion is that reserves actually have a stochastic nature and banks may earn a positive return on them.

In the sequel, we take the above discussion into account when assuming that the dynamics of the Treasuries and reserves are described by

$$
dT_t = r^T_t T_1 dt, \quad T_0 = b > 0
$$

$$
dR_t = R_t (\mu^R_t dt + \sigma^R_t dZ_t) dt + R_t \left\{ \int_{-1}^{\infty} x^R N^R(dt, dx^R) \right\}, \quad R_0 = r > 0
$$

respectively. Here $\mu^R$ is the rate of (positive) return earned by the bank and $\sigma^R$ is the volatility in the level of reserves. In order to have $R_t > 0$, we assume that $\sigma^R \Delta R_t > -1$ a.s. for all $t$. Next, we suppose that (8) holds and $\pi$ is the (possible time-dependent) proportion of the sum of Treasuries and reserves that is held in reserves. Taking our lead from (8), we assume that the stochastic dynamics of the sum of Treasuries and reserves, $W$, is given by

$$
dW_t = W_t \left\{ (r^T_t + \pi t (\mu^R_t - r^T_t)) dt + \pi \sigma^R_t dZ_t \right\} + k(t) dt + \pi t W_t - \int_{-1}^{\infty} x^R N^R(dt, dx^R)
$$

$$
W_{0-} = b + r = w > 0, \quad W_t = W_t^\mu = T_t^\mu + R_t^\mu \geq 0 \quad \text{for all } t \geq 0
$$

where the rate of depository consumption, $k(t)$, is the rate at which Treasuries and reserves are consumed by anticipated deposit withdrawals. Moreover, the solution of (9) may be given by

$$
W_t = W_0 \exp \left\{ \left[ \mu^R_t - 1/2 (\sigma^R_t)^2 \right] t + \sigma^R_t Z_t + \int_0^t \int_{|x|^2 < \infty} \{ \ln(1 + x^R) - x^R \} v(dx^R) ds \right\}
$$

$$
+ \int_0^t \int_\mathbb{R} \ln(1 + x^R) N^R(ds, dx^R)
$$

Copyright © 2007 John Wiley & Sons, Ltd.  
DOI: 10.1002/oca
The generator $G^w$ of the controlled process
\[ Q_t = \begin{bmatrix} s + t \\ W_t \end{bmatrix}, \quad t \geq 0, \quad Q_0 = q = \begin{bmatrix} s \\ w \end{bmatrix} \]
is given by
\[
G^w \varphi(q) = \frac{\partial \varphi}{\partial s} + \left( r_t^r (1 - \pi) + \mu_t^R \pi \right) w - k \frac{\partial \varphi}{\partial w} + \frac{1}{2} (\sigma_R^R)^2 w^2 \frac{\partial^2 \varphi}{\partial w^2}
+ \int_{-1}^\infty \left\{ \varphi(s, w + \pi w x^R) - \varphi(s, w) - \pi w x^R \frac{\partial \varphi}{\partial w}(s, w) \right\} v(dx^R)
\tag{10}
\]

2.1.3. Total unlevered assets. Suppose that the total unlevered assets, $A$, whose dynamics is given by (3), is a function, $g$, of $A$ and $W$ characterized by (6) and (9), respectively, i.e.
\[ A_t = g(A_t, W_t) \]
Then, for $\pi_t' = W_t / A_t$, the dynamics of $A$ may be represented by
\[
dA_t = A_t \left\{ [\mu(A_t, t) - f_t] dt + \sigma_t dZ_t \right\} + A_t \int_{-1}^\infty x \tilde{N}(dt, dx)
\tag{11}
\]
where
\[ \mu(A_t, t) = (1 - \pi_t') \mu_t^\Lambda + \pi_t' (r_t^r + \pi_t (\mu_t^R - r_t^T) - k(t)) \]
\[ \sigma_t dZ_t = (1 - \pi_t') \sigma_t^\Lambda dZ_t^\Lambda + \pi_t' \sigma_t^R dZ_t^R \]
\[ \int_{-1}^\infty x \tilde{N}(dt, dx) = \pi_t' \int_{-1}^\infty x^R N_t^R (dt, dx^R) + (1 - \pi_t') \int_{-1}^\infty x^\Lambda N_t^\Lambda (dt, dx^\Lambda) \]

We note that the standard Brownian motion, $Z$, and the compensated Poisson random measure, $\tilde{N}$, allow for a possible correlation between bank loan value, $A_t$, and the sum of Treasuries and reserves, $W$.

2.1.4. Risk-weighted assets. We consider risk-weighted assets (RWAs) that are defined by placing each on- and off-balance sheet item into a risk category. The more risky assets are assigned a larger weight. Figure 2 provides a few illustrative risk categories, their risk-weights and representative items.

As a result of the above, RWAs are a weighted sum of the various assets of the banks. In the sequel, we denote the risk weights on loans, Treasuries and reserves by $\omega^\Lambda$, $\omega^r$ and $\omega^R$, respectively. With regard to loans, we can identify a special risk weight denoted by $\omega(M_t) = \omega^L$ that is a decreasing function of current macroeconomic conditions, i.e.
\[ \frac{\partial \omega(M_t)}{\partial M_t} < 0 \]
This is in line with the procyclical notion that during booms, when macroeconomic activity increases, the risk weights will decrease. On the other hand, during recessions, risk weights may increase because of an elevated probability of default (PD) and/or loss given default (LGD) on loans (see, for instance, [38, 39]).

2.2. Bank capital

In this subsection, we discuss total bank capital, binding capital constraints and CARs.

2.2.1. Total bank capital. The bank’s total capital, \( C \), has the form

\[
C_t = C_t^{T1} + C_t^{T2}
\]

where \( C_t^{T1} \) and \( C_t^{T2} \) are Tier 1 and Tier 2 capital, respectively. Tier 1 (T1) capital is the book value of bank capital defined as the difference between the accounting value of the assets and liabilities. In our contribution, Tier 1 capital is represented at \( t^- \)’s market value of the bank equity, \( n_t E_t^- \), where \( n_t \) is the number of shares and \( E_t \) is the market price of the bank’s common equity at \( t \). Tier 2 (T2) capital consists of preferred stock and subordinate debt. Subordinate debt is subordinate to deposits and hence faces greater default risk. Tier 2 capital, \( O_t \), issued at \( t^- \) is represented by bonds that pay an interest rate, \( r^O \) (see, for instance, [40]).

2.2.2. Binding capital constraints. To reflect the book value property of regulatory capital and its market valuation sensitivity, we assume that at \( t^- \), the market value of equity and Treasuries determines the capital constraint to which the bank is subjected at \( t \). While there are several such constraints associated with Basel II, it is easy to show that the binding one is the total capital constraint. This constraint requires

\[
\rho_t = \frac{C_t}{a_t} \geq 0.08
\]

For the regulatory ratio of total capital to the weighted sum of loans, Treasuries and reserves denoted as, \( \overline{\rho} \), a capital constraint may be represented by

\[
\overline{\rho} [\omega^E \lambda_t + \omega^T T_t + \omega^R R_t] \leq n_t E_t^- + O_t
\]

(12)

As a result of (12), it is not necessary to differentiate between the relative cost of raising debt versus equity. Moreover, when maximizing profits, we consider the regulatory ratio of total capital to risk-weighted loans, \( \rho^f \), as an appropriate capital constraint. This means that we may set \( \omega^E = \omega(M_t) \) and \( \omega^T = \omega^R = 0 \) in (12) and express the binding capital constraint as

\[
\rho^f \omega(M_t) \lambda_t \leq n_t E_t^- + O_t
\]

(13)
The exact value of the regulatory ratio, $\rho^r$, may vary quite considerably from institution to institution (see, for instance, [16, 29]). In fact, subject to an appropriate choice for $\rho^r$, some banks may consider that equality in (13) implies an optimal choice of the investment in loans, $\lambda$, so that

$$\lambda^* = \frac{n_t E_t - O_t}{\rho^r \omega(M_t)}$$

2.2.3. Capital adequacy ratios. In this subsection, we discuss the dynamics of the CAR and regulatory thresholds for CARs. In the absence of a reliable procedure for determining $d_t$ from $\rho_t$ and $\Delta_t$ given by (1) and (6), respectively, we suppose that the value process $\rho_t$ at time $t$ of the CAR is a geometric Lévy process with dynamics

$$d\rho_t = \rho_t \{\mu^\rho dt + \sigma^\rho d\mathcal{L}_t\} + \xi \rho_t^{-1} \int_{\mathbb{R}} x^\rho N^\rho (dt, dx^\rho), \quad \rho_0 = z > 0$$

where $\xi$ is a constant such that $\xi x^\rho > -1$ a.s. $\nu$. An explicit solution of the SDE (14) has the form

$$\rho_t = \rho_0 \exp \left\{ \left[ \mu^\rho - \frac{1}{2} (\sigma^\rho)^2 - \xi \int_{\mathbb{R}} x^\rho v(dx^\rho) \right] t + \int_0^t \int_{\mathbb{R}} \ln(1 + \xi x^\rho) N^\rho (dt, dx^\rho) + \sigma^\rho d\mathcal{L}_t \right\}$$

As a rule, banks strive to maintain economic capital in excess of a regulatory minimum, $\rho^r$, with supervisors intervening otherwise. Basel II gives a precise description of the TRWAs to be used in the computation of the CAR without providing a description of the associated thresholds. Subsequently, we discuss a CAR threshold, $\rho^r$, in the context of supervisory auditing or review. If the bank undergoes such an audit at time $s + \tau$, the expected discounted difference between the actual CAR, $\rho$, and its regulatory threshold, $\rho^r$, is given by the performance criterion

$$J_{\tau}(s, z) := \mathbb{E}^{s, z} [\exp \{ - \delta (s + \tau) \} (\rho_{s + \tau} - \rho^r) \cdot \mathcal{X}_{\tau \in \mathbb{R}}]$$

where $\delta > 0$ is the discounting exponent. In situations where the value of $\rho$ is smaller than a regulatory threshold, $\rho^r$, regulators may pressure banks to increase the value of their CARs. This process may involve the withdrawal of insurance coverage, cease-and-desist orders, limits on asset growth and brokered deposits, prohibition of dividend payments and even bank closure. In the U.S.A., the prompt corrective action feature of the Federal Deposit Insurance Corporation Improvement Act (FDICIA) was implemented to improve capital-based incentives by making some of the aforementioned regulatory actions mandatory when CARs fall into certain capitalization categories (refer to the risk-based capital categories and supervisory risk subgroups in Figure 1). Furthermore, it is possible to find a fixed open set $\mathcal{E} \subset \mathbb{R} \times (0, \infty)$ that represents a region in which the bank is well or adequately capitalized and is known as the adequately capitalized category. In the spirit of the discussion above, we can consider the set

$$\tau_\mathcal{E} = \tau_\mathcal{E}(z, \omega) = \inf\{t > 0 : (t, \rho_t) \notin \mathcal{E}\}$$

to be the time instant at which $\rho_t = \rho^r$. In this case, for

$$X_t = \begin{bmatrix} s + t \\ \rho_t \end{bmatrix}$$

Copyright © 2007 John Wiley & Sons, Ltd.  
DOI: 10.1002/oca
we have that
\[
\begin{aligned}
\d X_t &= \left[ \begin{array}{c}
1 \\
\mu^\rho \\
\sigma^\rho
\end{array} \right] \d t + \left[ \begin{array}{c}
0 \\
\sigma^\rho \\
0
\end{array} \right] \d Z_t + \left[ \begin{array}{c}
0 \\
0 \\
\zeta \rho^\rho
\end{array} \right] \int_\mathbb{R} x^\rho \tilde{N}^\rho(\d t, \d x^\rho), \\
X_0 &= \left[ \begin{array}{c}
s \\
z
\end{array} \right]
\end{aligned}
\]

Also, the generator \( G^z \) of \( X_t \) is
\[
G^z \phi(s, z) = \frac{\partial \phi}{\partial s} + g z \frac{\partial \phi}{\partial z} + \frac{1}{2} z^2 \frac{\partial^2 \phi}{\partial z^2} + \int_\mathbb{R} \left\{ \phi(s, z + \zeta x^\rho) - \phi(s, z) - \zeta x^\rho \frac{\partial \phi}{\partial z} \right\} v(\d x^\rho)
\]

### 2.3. Liabilities

In our study, the only liability that we consider is deposits as it pertains to outstanding debt. In this case, the discussion on bank debt follows [1] (see, also, [4, 5]) rather closely. For \( A \) given by (11), we assume that \( Z_t = L_t \) in a time-invariant framework, where \( A_t = A, x_t = x, f_t = f, \sigma_t = \sigma \) and \( Z_t = Z \) (see, for instance, [1, 4, 5]). In essence this means that (11) becomes
\[
d A = A([\mu(A, t) - f] \d t + \sigma \d Z)
\]
where
\[
\mu(A, t) = (1 - \pi') \mu^A + \pi'(r^T + \pi(\mu^R - r^T) - k(t))
\]
\[
\sigma \d Z = (1 - \pi') \sigma^A dZ^A + \pi \sigma^R dZ^R
\]

#### 2.3.1. Deposits

For the pricing of bank debt, we suppose that the Treasury security, \( T \), described in Section 2.1.2 pays a constant Treasuries rate, \( r^T \). Furthermore, we assume that a non-negative coupon, \( c^+ \), per unit of time is continuously paid by the bank to its depositors. This situation persists until the bank declares default at the shareholder threshold, \( A^f \), or is closed by the regulatory authority at \( A' \). The value of the outstanding debt
\[
\Delta(A, x; A', A', \sigma, r^T, c^+, m, f) = \Delta(A, x; \ldots)
\]
may be represented by the solution of the non-linear ODE
\[
\frac{1}{2} \sigma^2 A^2 D_A \Delta(A) + (r^T - f) A D_A \Delta(A) - r^T \Delta(A) + c^+ + 1_{[A', A']} m(A - \Delta(A)) = 0
\] (17)

This solution will hold if either all agents are risk neutral or agents other than the regulator are always kept informed about the values of the assets and liabilities that are continuously traded. Furthermore, the economic boundary conditions for bank debt
\[
\lim_{A \to \infty} \Delta(A) = \frac{c^+}{r^T}
\]
(18)
\[
\lim_{A \to A'^+} \Delta(A) = \lim_{A \to A'^-} \Delta(A)
\]
(19)
\[
\lim_{A \to A'^+} D_A \Delta(A) = \lim_{A \to A'^-} D_A \Delta(A)
\]
(20)
\[
\Delta(A^f) = A^f
\]
(21)
must also be satisfied. By standard differential equation theory, the solution of (17) is given by

$$\Delta^{(1)}(A) = \frac{\epsilon^+}{r^+} + \beta^{(1)} A^p + \beta^{(2)} A^{p_2} \quad \text{for} A \geq A'$$

$$\Delta^{(2)}(A) = \frac{\epsilon^+}{r^+ + m} + \frac{m}{m + f} A + \epsilon^{(1)} A^{q_1} + \epsilon^{(2)} A^{q_2} \quad \text{for} A' \leq A \leq A'$$

In this regard, it is easy to verify that

$$p_{1,2} = \frac{\sigma^2 - 2(r^+ - f) \pm \sqrt{[\sigma^2 - 2(r^+ - f)]^2 + 8r^+ \sigma^2}}{2\sigma^2}$$

$$q_{1,2} = \frac{\sigma^2 - 2(r^+ - f) \pm \sqrt{[\sigma^2 - 2(r^+ - f)]^2 + 8(r^+ + m) \sigma^2}}{2\sigma^2}$$

where $p_1$ and $q_1$ are taken to be negative. As a consequence of the above, condition (18) implies that $\beta^{(2)} = 0$. For $\beta^{(1)}$, $\epsilon^{(1)}$ and $\epsilon^{(2)}$, conditions (19)–(21) lead to

$$\epsilon^{(1)} = \frac{A^{\#(c^+)} m c^+ p_1 / r^+ (m + r^+) + (m(1 - p_1) A^p / (m + f))}{(p_1 - q_1) A^{q_1} A^{q_2} - (p_1 - q_2) A^{s_1} A^{s_2}}$$

$$\epsilon^{(2)} = \frac{A^{\#(c^+)} m c^+ p_1 / r^+ (m + r^+) + (m(1 - p_1) A^p / (m + f))}{(p_1 - q_1) A^{q_1} A^{q_2} - (p_1 - q_2) A^{s_1} A^{s_2}}$$

$$\beta^{(1)} = \frac{mA^{\#(c^+)} q_1 A^q - q_1 p_1}{p_1 (m + f)} + \frac{\epsilon^{(1)} q_1 A^{q_1} - p_1}{p_1} + \frac{\epsilon^{(2)} q_2 A^{q_2} - p_1}{p_1}$$

2.4. Other banking items

In our study, the other banking items that we consider are subsidies, the total value of the bank and the value of capital (see, for instance, [1, 4, 5]). Throughout we assume that the dynamics of the unlevered assets value is given by (16).

2.4.1. Subsidies. We assume that the size of the government subsidy is proportional to the interest paid to the depositors, $r^+ c^+$, where $\tau$ is the subsidy rate. This implies that the value of the subsidy, $S$, satisfies the non-linear ODE

$$\frac{1}{2} \sigma^2 A^2 D_{AA} S(A) + (r^+ - f) A D_A S(A) - r^+ S(A) + \tau c^+ - 1_{[A', A']} m S(A) = 0$$
where the corresponding boundary conditions for subsidies are given by

\[
\lim_{A \to \infty} S(A) = \frac{\tau c^+}{r} \tag{30}
\]

\[
\lim_{A \to A^+} S(A) = \lim_{A \to A^-} S(A) \tag{31}
\]

\[
\lim_{A \to A^+} D_A S(A) = \lim_{A \to A^-} D_A \Delta(A) \tag{32}
\]

\[
S(A^*) = 0 \tag{33}
\]

In this case, for \(p_1, p_2, q_1\) and \(q_2\) given as in (24) and (25), the solutions of (29) have the form

\[
S^{(1)}(A) = \frac{\tau c^+}{r^+} + \tilde{\beta}^{(1)} A^{p_1} + \tilde{\beta}^{(2)} A^{p_2} \quad \text{for } A \geq A^* \tag{34}
\]

\[
S^{(2)}(A) = \frac{\tau c^+}{r^+ + m} + \frac{m}{m + f} + \tilde{\varepsilon}^{(1)} A^{q_1} + \tilde{\varepsilon}^{(2)} A^{q_2} \quad \text{for } A^* \leq A \leq A^* \tag{35}
\]

The economic boundary condition (30) implies that \(\tilde{\beta}^{(2)} = 0\) while conditions (31)–(33) suggest that

\[
\tilde{\varepsilon}^{(1)} = \frac{\tau c^+[mp_1 A^{q_2} / r^+ + (p_1 - q_2) A^{q_2}]}{(m + r^+)(p_1 - q_1) A^{q_1} A^{q_2} - (p_1 - q_2) A^{q_1} A^{q_2}} \tag{36}
\]

\[
\tilde{\varepsilon}^{(2)} = \frac{\tau c^+[mp_1 A^{q_2} / r^+ + (p_1 - q_1) A^{q_1}]}{(m + r^+)(p_1 - q_1) A^{q_1} A^{q_2} - (p_1 - q_2) A^{q_1} A^{q_2}} \tag{37}
\]

\[
\tilde{\beta}^{(1)} = \frac{\tilde{\varepsilon}^{(1)} q_1 A^{q_1 - p_1} + \tilde{\varepsilon}^{(2)} q_2 A^{q_2 - p_1}}{p_1} \tag{38}
\]

2.4.2. Total bank and capital value. We define the total value, \(v^T\), of the bank as

\[
v^T(A) = A + S(A) \tag{39}
\]

where \(A\) and \(S(A)\) are the asset value and subsidy value on coupon payments, respectively. In this context, (4) implies that the value of capital is given by

\[
C(A) = A + S(A) - \Delta(A) \tag{40}
\]

As a consequence, from Sections 2.3.1 and 2.4.1, we can conclude that

\[
C^{(1)}(A) = \frac{c^+ (\tau - 1)}{r^+} + A + (\tilde{\beta}^{(1)} - \beta^{(1)}) A^{p_1} \quad \text{for } A \geq A^* \tag{41}
\]

\[
C^{(2)}(A) = \frac{c^+ (\tau - 1)}{r^+} + \frac{f A}{m + f} + (\tilde{\varepsilon}^{(1)} - \varepsilon^{(1)}) A^{q_1} + (\tilde{\varepsilon}^{(2)} - \varepsilon^{(2)}) A^{q_2} \quad \text{for } A^* \leq A \leq A^* \tag{42}
\]
3. OPTIMAL AUDITING IN THE BANKING INDUSTRY

In the ensuing section, we discuss random and non-random auditings for reserve, asset and capital requirements. The main results in Sections 3.1 and 3.3 can be verified by using standard techniques in the stochastic analysis of jump diffusions. The main banking indicators that we discuss in the sequel are the sum of the Treasuries and reserves, $W$, and CAR, $\rho$.

3.1. Random auditing: reserve requirements

Auditors ought to be able to access information about the volume of (Treasuries and) reserves that the bank holds as a provision for deposit withdrawals. This auditing criterion is called the reserve requirement. Both auditors and banks are interested in establishing the optimal level of reserves on demand deposits that the bank must hold. By setting a bank’s individual optimal level of reserves, auditors assist in mitigating the costs of financial distress.

In the sequel, we discuss how model (9) for Treasuries and reserves built in Section 2.1.2 may be used to obtain an important result in the stochastic optimization of the reserve requirement (see, for instance, [41]). From the point of view of the bank owner, we may pick any utility function, $U : [0, \infty) \to \mathbb{R}$, but, in our case, the choice of power utility

\[
U(k) = \frac{1}{\alpha} k^{2} \quad \text{with } 0 < \alpha < 1
\]

leads to an analytic solution. In this case, the performance criterion is the objective functional given by

\[
J_t(\pi, k) = \mathbb{E}^w \left[ \int_0^\infty \exp \left\{ -\delta(s+t) \right\} U(k) \, dt \right]
\]

(44)

In order to determine the optimal reserve allocation, $\pi^*$, and depository consumption, $k^*$, we consider the set of admissible controls

\[
\mathcal{A} = \{ u(t) = (\pi_t, k(t)) : (9) \text{ has a unique strong solution and (44) has a finite value} \}
\]

(45)

Also, the value function is given by

\[
V(t, w) = \sup_{(\pi, k) \in \mathcal{A}} \mathbb{E}^w \left[ \int_0^\infty \exp \left\{ -\delta(s+t) \right\} \frac{1}{\alpha} k^2(t) \, dt \right]
\]

(46)

The optimal auditing problem for Treasuries and reserves may be formally stated as follows.

**Problem 3.1 (Optimal bank reserves and depository consumption rate)**

Suppose that the sum of the Treasuries and reserves, $W$, utility, $U$, and performance criterion, $J$, and admissible class of control laws, $\mathcal{A} \neq \emptyset$, are described by (9), (43)–(45), respectively. In this case, determine $V(t, w)$ in (46) and the optimal control law $(\pi^*, k^*)$, if it exists.

The main optimization result for bank reserve allocation and depository consumption follows below.

**Theorem 3.2 (Optimal bank reserves and depository consumption rate)**

Suppose that the sum of the Treasuries and reserves, $W$, terminal utility, $U$, performance criterion, $J$, set of admissible controls, $\mathcal{A}$, and value function, $V$, are characterized by (9), (43)–(46),
respectively. Furthermore, assume that the generator $G^w$ of $Q$ is given by (10). Then the optimal bank reserve allocation is $\pi^*$ which solves the integral equation

$$\Phi(\pi^*) = \mu^R - r^x - \sigma^R \pi(1 - x) - \int_{-1}^{\infty} \{1 - (1 + \pi x^R)^{\gamma - 1}\} x^R \nu(dx) = 0, \quad \pi^* \in (0, 1]$$

(47)

If we assume that

$$K = \frac{1}{x} \left[ \frac{1}{1 - x} \left( \delta - x(r^x - 1) + \mu^R \pi^* \right) + \frac{1}{2} \sigma(1 - x) \sigma^R \pi^* \right]$$

$$- \int_{R} \left\{ (1 + \pi^* x)^{\gamma - 1} - 1 - x \pi^* x \right\} \nu(dx) \right]^{\gamma - 1}$$

(48)

then the optimal depository consumption rate is given by

$$k^* = (xK)^{1/\gamma - 1} w$$

(49)

Proof

The proof is outlined in Appendix A of Section A.2.

3.2. Random auditing: asset requirement

In this subsection, we discuss the low contact condition and its relation with the existence of a solution to the asset threshold problem.

3.2.1. Low contact condition. Since the closure rule, $A^s$, is chosen by the bank shareholders, $A^s$ may be obtained via the low contact condition

$$D_A C(A^s) = 0$$

(50)

that, in turn, leads to

$$\frac{f A^s}{m + f} + (\bar{c}^{(1)} - \bar{c}^{(1)}) q_1 A^{s_1} + (\bar{c}^{(2)} - \bar{c}^{(2)}) q_2 A^{s_2} = 0$$

(51)

We recall from Sections 2.3 and 2.4 that the closure rule, $A^r$, implies the minimum capital adequacy requirement that triggers bank closure and reorganization if the situation $A < A^r$ is found during a regulatory audit. In order to comply with empirical norms, the model of the regulatory regime requires that the total value of the bank assets relative to the face value of the liabilities when $A = A^r$ should be less than unity. For all $A \geq A^r$, a requirement is that the regulator chooses $A^r$ in such a manner that the shareholders become indifferent with respect to the risk taken by the bank, $\sigma$. More formally, this yields

$$\frac{\partial C \bar{c}^{(1)}(A)}{\partial \sigma^2} = 0$$

(52)

In general, however, the threshold $A^r$ that satisfies this condition will depend on $\sigma$. 

3.2.2. Existence of a solution to the asset threshold problem. The following existence result related to the asset threshold problem comes directly from [1] and provides information about the thresholds, $A^r$ and $A^s$.

Theorem 3.3 (Existence of solution ($A^r$, $A^s$) to asset threshold problem)
Suppose that the low contact condition (50) holds and that $u = A^r / A^s$
Then there exists a solution $(A^r, A^s)$ with the property that $0 < A^s < A^r$ if and only if
$$ q_1 (p_1 - p_2) u q_2 - q_2 (p_1 - q_1) u q_1 + (m / r^T) p_1 (q_1 - q_2) $$
$$ - q_1 (p_1 - q_2) u q_1 + q_2 u q_2 + (m / r^T) p_1 (q_1 u q_1 - q_2 u q_2) $$
$$ - p_1 (q_1 - q_2) u q_1 + q_2 u q_2 - m p_1 u [(1 - q_1) u q_1 - (1 - q_2) u q_2] = 0 $$
has a solution $u > 1$. Moreover, $A^r$ and $A^s$ are linear in the coupon, $c^+$, and the subsidy rate, $\tau$. Also, $A^r$ and $A^s$ do not depend on $A$.

Proof
The outline of the proof is provided by Appendix C.

3.3. Non-random auditing: capital requirements
In this section, our analysis makes use of the SDE for CARs given by (14). In the sequel, this ratio plays a central role in an optimal auditing time problem that, from the bank owner’s viewpoint, provides an indication of when auditing by the regulator would yield the most favorable outcome.

3.3.1. An optimal auditing time problem for CARs. In the analysis that follows, we will concentrate our efforts on finding a precise formula for the value function, $V(s, z)$, for which the performance criterion, $J$, is defined by (15). Here, the set of admissible controls is given by
$$ \mathcal{A}^{(1)} = \{(s, z) : (14) has a unique strong solution and (15) has a finite value\} $$
and the value function has the form
$$ V^{(1)}(s, z) = \sup_{(s, z) \in \mathcal{A}^{(1)}} \mathbb{E}^{s, z} [\exp{-\delta(s + \tau)}] (\rho_{\tau} - \rho^r) \cdot \mathcal{X}_{\tau < \infty} $$
We are now in a position to state the optimal auditing time problem that we solve in the sequel.

Problem 3.4 (Optimal auditing time problem)
Suppose that the SDE for the $\rho$-dynamics is given by (14) and the performance criterion, $J$, is represented by (15). In this case, determine $V^{(1)}(s, z)$ in (55) and the optimal auditing time $\tau^*$, if it exists.

3.3.2. Solution to the optimal auditing time problem. In this subsection, we derive a solution for Problem 3.4 in the case where the dynamics of $\rho$ is characterized by (14).
Theorem 3.5 (Solution to the optimal auditing time problem)
Suppose that the ρ-dynamics is described by (14), \( q < \delta \) and
\[
q > \frac{1}{2} \sigma^2 + \xi \int_\mathbb{R} x \nu(dx) \quad \text{and} \quad 2q - 2\delta + \sigma^2 + \xi^2 \int_\mathbb{R} x^2 \nu(dx) < 0
\]
hold. Then with \( \kappa > 1 \), \( z^* \) and \( K^{(1)} \) given by
\[
\delta - \kappa q - \frac{1}{2} \sigma^2 \kappa (\kappa - 1) = \int_{-1}^{\infty} \{(1 + \xi x)^\kappa - 1 - \kappa \xi x\} \nu(dx), \quad z^* = \frac{\kappa}{\kappa - 1} \rho^r \quad \text{and} \quad (56)
\]
respectively, the function
\[
\varphi(s, z) = \begin{cases} 
\exp\{-\delta s\} K^{(1)} z^k, & 0 < z < z^* \\
\exp\{-\delta s\} K^{(1)} (z - \rho^r), & z^* < z
\end{cases}
\]
corresponds to the value function, \( V^{(1)}(s, z) \), given by (55). Furthermore, we have that \( \tau^* = \tau_G \) is an optimal auditing time where
\[
\mathcal{D} = \{ (s, z) : 0 < z < z^* \}, \quad z^* > \frac{\delta}{\delta - q} \rho^r, \quad \tau_G = \inf\{ t > 0 : (t, z) \notin \mathcal{D} \} \quad (58)
\]

Proof
The outline of the proof is provided by Appendix B in Section B.3.

4. ANALYSIS OF THE MAIN ECONOMIC ISSUES

In accordance with the dictates of the Basel II, the models of bank items constructed in this paper are related to the methods being used to assess the riskiness of bank portfolios and their minimum capital requirement (see [12, 15]).

4.1. Stochastic model for banks

In this subsection, we analyze aspects of the bank items such as assets, capital and liabilities that are introduced in Section 2.

4.1.1. Assets. The opportunity for bank investment in loans is discussed in Section 2.1.1, where the loan demand is Hicksian and described in terms of macroeconomic activity in the loan market. Banks experience shocks that affect the value of the loan demand, \( \Lambda \), when the minimum capital requirements for Basel II are calculated by using risk-weighted assets. In the Hicksian case typified by (5), these responses are usually sensitive to macroeconomic conditions that are related to the term \( l_2 M_t \). The discussion in Section 2.1.1 leads to a definition of the elasticity of demand, \( d^c \), of the form
\[
d^c = \frac{l_1 r^A}{k_t}
\]
It is well known that if the bank is perfectly competitive, \( d^c \) will tend to \( \infty \). This, of course, is not the case in our imperfectly competitive paradigm.

The bank’s investment in loans may yield substantial returns but may also result in loan losses. Loan defaults are generally independent of the capital adequacy paradigm that is chosen. In this regard, empirical evidence suggests that better macroeconomic conditions reduce the loan default rate and thus the loan marginal cost (see the discussion on the procyclicality of loan losses in, for instance, [38, 39, 42]). In line with this reality, our dynamic banking loan model in (6) allows for loan losses and their partial provisioning. The related default risk may be modeled as a compound Poisson process, where \( N \) is a Poisson process, with a deterministic frequency parameter, \( \phi(t) \). Here \( N \) is stochastically independent of the Brownian motion, \( Z \), given in (6). Furthermore, we introduce the value of loan losses as

\[
L(M_t, t) = r^d(M_t)\lambda_t
\]

where \( L \) is independent of \( N \). Also, we may assume that the default or loan loss rate, \( r^d \in [0, 1] \), increases when macroeconomic conditions deteriorate according to

\[
0 \leq r^d(M_t) \leq 1, \quad \frac{\partial r^d(M_t)}{\partial M_t} < 0
\]

As is the case with the relationship between profit and macroeconomic activity, the above description of the loan loss rate is consistent with empirical evidence that suggests that bank losses on loan portfolios are correlated with the business cycle under any capital adequacy regime (see, for instance, [38, 39, 42, 43]). Furthermore, it may be that the provision made by the bank for loan losses takes the form of a continuous contribution that can be expressed as

\[
[1 + \theta(s)]\phi(s)E[P_t(l)]
\]

where \( \theta \) is a loading term dependent on the level of credit risk, \( \theta(t) \geq 0 \) and \( P_t \) is the actual provision for loan losses. This means that if the bank suffers a loan loss of \( \lambda = l \) at time \( t \), the provisions, \( P_t(l) \), covers these losses.

Several interesting contributions have led to the choice of representation (9) for the dynamics of the sum of the Treasuries and reserves. Amongst these is a paper by Chan (see [44]) that treats the case where the Lévy process decomposition of general assets is a lot finer than is the case here.

4.1.2. Bank capital. Despite the analysis in Section 2.2, bank capital is notoriously difficult to define, monitor and measure. In this regard, for instance, the measurement of equity depends on how all of a bank’s financial instruments and other assets are valued. In general, the modeling of the shareholder equity component of bank capital, \( E \), is underpinned by the following two observations. In the first place, it is meant to reflect the nature of the book value of equity and, secondly, to recognize that the book and market value of equity is highly correlated.

Under Basel II, bank capital requirements have replaced reserve requirements (see Section 2.1.2) as the main constraint on the behavior of banks. A first motivation for this is that bank capital has a major role to play in overcoming the moral hazard problem arising from asymmetric information between banks, creditors and debtors. Also, bank regulators require capital to be held to protect themselves against the costs of financial distress, agency problems and the reduction of market discipline caused by the safety net.
Section 2.2.2 suggests that a close relationship exists between bank capital holding and macroeconomic activity in the loan market. As was mentioned before, Basel II dictates that a macroeconomic shock will affect the loan risk weights in the CAR. In general, a negative (positive) shock results in the tightening (loosening) of the capital constraint given by (13). As a consequence, in terms of a possible binding capital constraint, banks are free to increase (decrease) the loan supply when macroeconomic conditions improve (deteriorate). On the other hand, if the risk weights are constant, a shock does not affect the loan supply but rather results in a change in the loan rate when the capital constraint binds. It is not always true that Basel II risk-sensitive weights lead to an increase (decrease) in bank capital when macroeconomic activity in the loan market increases (decreases). A simple explanation for this is that macroeconomic conditions do not necessarily only affect loan demand but also influences the total capital constraint from (13). Furthermore, banks do not necessarily need to raise new capital to expand their loan supply, since a positive macroeconomic shock may result in a decrease in the RWAs with a corresponding increase in CARs (compare (1)). Similarly, banks are not compelled to decrease their capital when the loan demand decreases since the capital constraint usually tightens in response to a negative macroeconomic shock. A further complication is that an improvement in economic conditions may result in an increase in the loan demand and, as a consequence, an increase in the probability that the capital constraint will be binding. Banks may react to this situation by increasing capital to maximize profits (compare the definition of the return on equity (ROE)). Our main conclusion is that bank capital is procyclical because it is dependent on fluctuations in loan demand which, in turn, is reliant on macroeconomic activity.

4.1.3. Liabilities. An additional issue in the modeling of debt relates to the possibility that unanticipated deposit withdrawals, $u$, will occur. By way of making provision for these withdrawals, the bank is inclined to hold Treasuries, $T$, and reserves, $R$, that are very liquid. In our contribution, we assume that $u$ may be associated with the probability density function, $f(u)$, that is independent of time. In this regard, we may suppose that the unanticipated deposit withdrawals have a uniform distribution with support $[\underline{u}, \overline{u}]$ so that the cost of liquidation, $c^l$, or additional external funding is a quadratic function of the sum of reserves and Treasuries, $W = R + T$. In addition, for any $t$, if we have that

$$u > W_t$$

then bank assets are liquidated at some penalty rate, $r^p_t$. In this case, the cost of deposit withdrawals is

$$c^u(W_t) = r^p_t \int_{W_t}^\infty [u - W_t] f(u) du = \frac{r^p_t}{2\overline{u} - W_t} [\overline{u} - W_t]^2$$

We note that condition (18) in Section 2.3.1 holds because, under the current assumptions, default becomes irrelevant as the value of $A$ increases and the value of debt approaches the value of the capitalized coupon and therefore the value of risk-free debt. In addition, (19) and (20) are smooth pasting conditions, while condition (21) and the last term in (17) guarantee that in the case of default or closure the value of debt is equal to the asset value of the bank. On the other hand, (30) in Section 2.4.1 holds because of the irrelevance of default as $A$ grows and the value of the subsidy, $S$, approaches its riskless capitalized present value. Conditions (31) and (32) are the common smooth pasting conditions, and condition (33) reflects the loss of the (tax-) subsidy.
benefits as far as the current bank owners are concerned in the event that the bank declares default or faces closure.

4.2. Optimal auditing in the banking industry

In this subsection, we discuss some of the issues related to the optimal auditing problem presented in Section 3.

4.2.1. Random auditing: reserve requirements. As was mentioned before, we define the rate of depository consumption as the rate at which Treasuries and reserves are consumed by the taking, holding and anticipated withdrawal of deposits. Here, a typical bank owner has to make decisions about deposit taking via the fixing of costs related to cheque clearing and bookkeeping, the holding of deposits by means of the choice of the deposit and coupon rate and anticipated withdrawals via the provisioning provided by Treasuries and reserves. We note that we are dealing with the bank owner’s utility in (43) where the optimal rate of depository consumption in Theorem 3.2 is expressed almost exclusively in terms of the components of the Treasuries and reserves. For mutual banks, of course, the aforementioned utility corresponds with the depositor’s utility because they are also the owners of the bank.

The formulation of Theorem 3.2 of Section 3.1 suggests that \( V \) and Theorem 3.5 of [45] can be used to derive the associated HJB equation for optimal control of jump diffusions. In addition, certain verification theorems claim that if the objective function, \( V \), is smooth and the related HJB equation has a smooth solution, \( \tilde{V} \), then under certain regularity conditions, \( V = \tilde{V} \). We are able to appeal to the theory of viscosity solutions in those cases for which a smooth solution does not exist.

The optimal consumption rate from (49) in Theorem 3.2 suggests that \( k^* \) is directly proportional to the sum of the Treasuries and reserves, \( w = T_0 + R_0 \) at \( t = 0 \). Here the proportionality constant \((xK)^{1/z-1}, 0<z<1\), is related to the discounting exponent, \( \delta \), the bank reserve allocation, \( \pi^* \), the dynamics of the sum of Treasuries and reserves, \( dW \), as well as, by definition, the loans, \( \Lambda \). As a consequence, the associated control law would seem to suggest that as the initial value of the sum of Treasuries and reserves, \( w \), increases (decreases) the optimal consumption rate, \( k^* \), will also increase (decrease). In essence, this increase may be due to an increase (decrease) in the deposit rate and resulting depository activity.

A strong correlation has been established between reserve requirements and bank default and closure. In order to monitor the fluctuations in the reserves, in practice, a regular liquid reserve requirement report compiled by the individual bank may be audited along with its financial statements and relevant information. This process may result in a consolidated liquid reserve requirement report that is authored by the auditors. Effective legal reserve requirements may hamper the private capital market’s ability to price bank deposits. In the model developed in [46], the market has less information about bank assets than the banks have, and a bank can therefore signal its superior information through its choice of excess reserves. Mandatory reserves can inhibit such signaling and therefore result in inefficient deposit pricing.

4.2.2. Random auditing: asset requirement. Generally, the critical \( A' \) that satisfies condition (52) will depend on the level of \( \sigma \). One interpretation is that the regulator announces \( A' (\sigma) \) and then the bank picks \( \sigma \). Thus, the practical default and closure rules implied by the model corresponds well with the regulatory rules imposed by many OECD countries, for example, by the EU capital
adequacy directives. These specify that the bank’s equity requirements are directly proportional to its value at risk which in our framework is proportional to $\sigma$.

We conclude from Theorem 3.3 that there exists $c^+$-independent parameters $j_1$, $k_1$, $\tilde{k}_1$, $k_2$ and $\tilde{k}_2$, such that

$$\Delta^{(1)}(A) = \frac{c^+}{r^T} \left[ 1 - j_1 \left( \frac{A}{c^+} \right)^{p_1} \right] \text{ for } A \geq A'$$

(60)

$$\Delta^{(2)}(A) = \frac{mA}{m+f} + \frac{c^+}{r^T} \left[ \frac{r^T}{m+r^2} - k_1 \left( \frac{A}{c^+} \right)^{q_1} - k_2 \left( \frac{A}{c^+} \right)^{q_2} \right] \text{ for } A^s \leq A \leq A'$$

(61)

$$v^{T(1)}(A) = A + \frac{c^+}{r^T} \left[ \tau - a_1 \left( \frac{A}{c^+} \right)^{p_1} \right] \text{ for } A \geq A'$$

(62)

$$v^{T(2)}(A) = A + \frac{c^+}{r^T} \left[ \frac{\tau r^T}{m+r^2} - \tilde{k}_1 \left( \frac{A}{c^+} \right)^{q_1} - \tilde{k}_2 \left( \frac{A}{c^+} \right)^{q_2} \right] \text{ for } A^s \leq A \leq A'$$

(63)

$$C^{(1)}(A) = A - \frac{c^+ (1-\tau)}{r^T} \text{ for } A \geq A'$$

(64)

$$C^{(2)}(A) = \frac{f}{(f+\tau)} A - \frac{c^+}{r^T} \left[ (1-\tau) - (\tilde{k}_1 - k_1) \left( \frac{A}{c^+} \right)^{q_1} - (\tilde{k}_2 - k_2) \left( \frac{A}{c^+} \right)^{q_2} \right]$$

(65)

$$\text{for } A^s \leq A \leq A'$$

4.2.3. Non-random auditing: capital requirements. An important observation emanating from Section 4.2.2, about the connection between asset and capital requirements, is that

$$\rho^{L(1)}(A) = 1 - \frac{c^+ (1-\tau)}{Ar^T} \text{ for } A \geq A'$$

(66)

$$\rho^{L(2)}(A) = \frac{f}{(f+\tau)} A - \frac{c^+}{r^T} \left[ (1-\tau) - (\tilde{k}_1 - k_1) \left( \frac{A}{c^+} \right)^{q_1} - (\tilde{k}_2 - k_2) \left( \frac{A}{c^+} \right)^{q_2} \right]$$

(67)

These formulas follow directly from (64) and (65) and the asset value, $A$. This reinforces the fact that $A$ plays an important role as a constituent of the CAR, $\rho$, defined by (1). A further implication of this is that $A^s$ and $A'$ define the standard for minimum capital adequacy in an implicit way.

The solution to the optimal auditing time problem presented in Theorem 3.5 of Section 3.3 has a few interesting ramifications for capital adequacy regulation. From (56), we note for the optimality exponent $\kappa > 1$ and $\rho^r > 0$ that $z^*$ must have a positive value that is greater than the value of $\rho^r$ by a specific proportionality constant. At optimality, (56) and (57) allow a direct comparison between $\kappa$ on the one hand and the discounting exponent, $\delta$, and the coefficient, $q$, of the second term of
the generator, $G^z$, on the other. More specifically, for an optimal CAR, $z^*$, we have that

$$\frac{\kappa}{\kappa - 1} \geq \frac{\delta}{\delta - \varrho}$$

In addition, (57) and (58) seem to imply, for $(t, z) \in D$, that $z^k$ is an important index for auditing purposes. This may suggest that banks with a suboptimal CAR value should be more concerned with the CAR value itself rather than its relationship with the regulatory ratio, $\rho'$. By contrast, for $(t, z) \notin D$, the value of $\rho'$ now plays a significant role. This may have something to do with the function of bank capital holdings as both a buffer against insolvency and an indicator of profitability via the ROE indices. Finally, (58) intimates that the optimal CAR, $z^*$, is bounded away from the regulatory threshold, $\rho'$, by a constant factor that depends on the discounting exponent, $\delta$, and the coefficient of the second term, $\varrho$, of the generator, $G^z$.

The optimization problem in Theorem 3.5 of Section 3.3 only makes sense if the bank owner can decide when to be audited. In particular, the formulas in (56) only yield the correct value function, $V^{(1)}(s, z)$, providing auditing actually happens at $z = z^*$. In particular, the value function has the power form $z^k$ before auditing occurs and the linear form $z^* - \rho'$ after auditing occurs.

5. CONCLUDING REMARKS

The reliability, transparency and quality of audited financial statements are critical to the efficient allocation of resources by role players in the banking industry. Auditors can benefit greatly by the employment of sound auditing models during the auditing procedure.

In Theorem 3.2, we considered the decision about the allocation of reserves held by banks. The important that we can switch between Treasuries and reserves at will. The aforementioned allocation strategy is instrumental in establishing guidelines that inform the optimal volume of (Treasuries and) reserves required during a random audit. Furthermore, Theorem 3.5 establishes an optimal auditing time that is related to an optimal CAR level. As a consequence of this, we are able to add to the debate about one of the major shortcomings of Basel II, viz., the characterization of regulatory thresholds like closure and corrective action thresholds. In the main, banking items that constitute the optimal reserve allocation and auditing time problems are described by jump diffusion models. Finally, in Section 4, we discuss some of the economic issues arising from the analysis of stochastic dynamic models of banking items and the optimization of the auditing process.

Several interesting questions related to auditing in the banking industry remain open. Amongst these is the development of multi-criteria decision-aid techniques for auditors. Also, in our paper, the regulatory strategy is strongly related to incentive compatibility. Also, we require that risk shifting incentives are completely eliminated by the closure rules. In a more general framework than the one used in our paper, regulators should be able to take into account their monitoring costs and deadweight costs of deposit insurance and bank reorganization. Furthermore, future modeling should bear in mind that changes to the riskiness of a bank’s operations also influence other parameters such as its asset value. If this effect is very strong, then eliminating risk-shifting incentives to deviate from the overall value-maximizing investment choice would be most desirable. In general, the implementation of any regulatory strategy will require that it is incentive compatible for the banks shareholders not to alter the investment strategies on which regulation is based.
OPTIMAL AUDITING IN THE BANKING INDUSTRY

The ensuing appendices have been compiled upon the request of an anonymous referee. It is intended to make the technical aspects of the main results of the paper more accessible to the reader.

In this section, we present the background results needed to prove Theorems 3.2, and 3.5. The proof of the former theorem utilizes information from [41] and Theorem 3.5 of [45]. On the other hand, the proof of Theorem 3.5 makes use of Theorems 2.2 and 2.3 of Chapter 2 in [45] which are presented in Appendices B.1 and B.2, respectively. Furthermore, the proof of Theorem 3.2 is reliant on the analysis in [1].

APPENDIX A: OPTIMAL CONTROL OF JUMP DIFFUSIONS

We consider an open set \( \mathcal{S} \subset \mathbb{R}^n \) which is called a solvency region and let \( Y_t = Y_t(u) \) be a stochastic process in \( \mathbb{R}^n \) of the form

\[
    dY_t = b(Y_t, u(t)) \, dt + \sigma(Y_t, u(t)) \, dZ_t + \int_{\mathbb{R}^n} \zeta(Y_t -, u(t -), x) \, \tilde{N}(dx, dt), \quad Y_0 = y \in \mathbb{R}^n
\]

(A1)

where \( b: \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \), \( \sigma: \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^{n \times m} \) and \( \zeta: \mathbb{R}^k \times \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^{n \times l} \) are functions and \( \mathcal{U} \subset \mathbb{R}^k \) is a given set. The process \( u(t) = u(t, \omega): [0, \infty) \times \Omega \to \mathcal{U} \) is the control process that is assumed to be cadlag and adapted. In this case, we call \( Y_t = Y_t(u) \) a controlled jump diffusion. In the sequel, we consider a performance criterion \( J = J(u)(y) \) of the form

\[
    J(u)(y) = \mathbb{E}^y \left[ \int_0^{\tau_{\mathcal{S}}} f(Y_t, u(t)) \, dt + g(Y_{\tau_{\mathcal{S}}}) \cdot \mathcal{X}_{[\tau_{\mathcal{S}} < \infty]} \right]
\]

where

\[
    \tau_{\mathcal{S}} = \tau_{\mathcal{S}}(z, \omega) = \text{inf}\{t > 0: Y_t(u) \notin \mathcal{S}\}
\]

is the bankruptcy time and \( f: \mathbb{R}^n \to \mathbb{R} \) and \( g: \mathbb{R}^n \to \mathbb{R} \) are continuous functions. We say that the control process \( u \) is admissible and write \( u \in \mathcal{A} \) if (A1) has a unique strong solution \( Y_t \) for all \( y \in \mathcal{S} \) and

\[
    \mathbb{E}^y \left[ \int_0^{\tau_{\mathcal{S}}} f(Y_t, u(t)) \, dt + g(Y_{\tau_{\mathcal{S}}}) \cdot \mathcal{X}_{[\tau_{\mathcal{S}} < \infty]} \right] < \infty
\]

The stochastic control problem is the following.

Find \( V(y) \) and an optimal control \( u^* \in \mathcal{A} \) such that

\[
    V(y) = \sup_{u \in \mathcal{A}} J(u)(y) = J(u^*)(y)
\]

We know that under mild conditions it suffices to consider Markov control, i.e. controls \( u(t) \) of the form

\[
    u(t) = u_0(Y_t -)
\]
for some function $u_0 : \mathbb{R}^n \to \mathcal{U}$. We only consider Markov controls and write $u(t) = u(Y_{t-})$. Note that if $u = u(y)$ is a Markov control then $Y_t = \hat{Y}^{(u)}_t$ is a Lévy diffusion with generator

$$G \varphi(y) = G^u \varphi(y) = \sum_{j=1}^n b_j(y, u(y)) \frac{\partial \varphi}{\partial y_j}(y) + \frac{1}{2} \sum_{j,k=1}^n (\sigma \sigma^T)_{jk}(y, u(y)) \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(y)$$

$$+ \sum_{k=1}^l \int_{\mathbb{R}} \left\{ \varphi(y + \zeta^{(k)}(y, u(y), x_k)) - \varphi(y) - \nabla \varphi(y) \cdot \zeta^{(k)}(y, u(y), x_k) \right\} v_k(dx_k)$$

\[\text{(A2)}\]

**A.1. Hamilton–Jacobi–Bellman for optimal control of jump diffusions**

Suppose that there exists a function $\varphi : \mathcal{F} \to \mathbb{R}$ such that

(A.1.1) $\varphi \in C^2(\mathcal{F}) \cap C(\overline{\mathcal{F}})$,

(A.1.2) $Y(\tau_{\mathcal{F}} \in \partial \mathcal{F}$ a.s. on $\{\tau_{\mathcal{F}} < \infty\}$ and for all $u \in \mathcal{A}$ we have

$$\lim_{t \to \tau_{\mathcal{F}}} \varphi(Y_t) = g(Y_{\tau_{\mathcal{F}}}) \cdot \mathcal{L}[\tau_{\mathcal{F}} < \infty]$$

(A.1.3) For all $u \in \mathcal{A}$ and all $\tau \in \mathcal{F}$ we have that

$$\mathbb{E}^y \left[ |\varphi(Y_{\tau})| + \int_0^{\tau_{\mathcal{F}}} \left\{ |G \varphi(Y_t)| + \sigma^T(Y_t) \nabla \varphi(Y_t) |^2 + \sum_{k=1}^l \left\{ \int_{\mathbb{R}} |\varphi(Y_t) + \zeta^{(k)}(Y_t, u(t), x_k) - \varphi(Y_t)|^2 v_k(dx_k) \right\} dt \right] < \infty$$

\[\text{(A3)}\]

(A.1.4) $\{\varphi(Y_{\tau}) : \tau \in \mathcal{F}\}$ is uniformly integrable for all $u \in \mathcal{A}$ and $y \in \mathcal{F}$.

If (A.1.1)–(A.1.4) hold, then, for all $y \in \overline{\mathcal{F}}$, we have that

$$\varphi(y) \geq V(y)$$

Moreover, assume that for each $y \in \overline{\mathcal{F}}$ there exists $v = \hat{u}(y) \in \mathcal{U}$ such that

(A.1.5) $G^{\hat{u}(y)} \varphi(y) + f(y, \hat{u}(y)) = 0$

(A.1.6) $\{\varphi(Y_{\tau}^{(u)}) \}_{\tau \leq \tau_{\mathcal{F}}}$ is uniformly integrable.

Suppose $u^*(t) := \hat{u}(Y_{t-}) \in \mathcal{A}$. Then $u^*$ is an optimal control and

$$\varphi(y) = V(y) = J^{(u^*)}(y) \quad \text{for all } y \in \mathcal{F}$$

**A.2. Proof of Theorem 3.2**

The main thrust of the proof is to show that conditions (A.1.1)–(A.1.6) in Appendix A.1 are satisfied. In order to accomplish this, we employ a Hamilton–Jacobi–Bellman equation (HJBE)
approach for optimal control of jump diffusions (see [41] and Theorem 3.1 of Chapter 3 in [45]).
For the generator of $Q$ given by (10) we choose
$$\varphi(q) = \varphi(s, w) = \exp\{-\delta s\}K w^x$$
where $K$ is given by (48). In this case, the form of (10) leads to
$$g(w, \pi) = G w^x + f(w, u)$$
(A4)
where, for a constant $\theta$, we have that
$$g(w, \pi) = -K w^x + (\mu^T(1-\pi) + \mu R \pi)w - k)K x w^{x-1} + K \frac{1}{2} (\sigma R)^2 \pi^2 w^x x(\pi - 1) w^{x-2}$$
+ $K w^x \int_{-1}^{\infty} \{(1 + \pi x R)^2 - 1 - x\pi x R\}v(dx R) + \frac{w^x}{x}$

In this case, $g(k, \pi)$ is concave in $(k, \pi)$ and attains a maximum in the case where
$$\frac{\partial g}{\partial \pi} = (\mu^T R - \mu R)K x w^{x-1} + K \frac{1}{2} (\sigma R)^2 \pi x(\pi - 1) w^{x-2}$$
+ $K w^x \int_{-1}^{\infty} \{(1 + \pi x R)^2 - 1 - x\pi x R\}v(dx R) = 0$
(A5)
and
$$\frac{\partial g}{\partial k} = -K x w^{x-1} + k^{x-1} = 0$$
(A6)
From (A5) and (A6) we obtain (47) and (49), respectively.

APPENDIX B: OPTIMAL STOPPING OF JUMP DIFFUSIONS

We consider an open set $S \subset \mathbb{R}^n$ which is called a solvency ratio and let $Y_t$ be a jump diffusion
in $\mathbb{R}^n$ of the form
$$dY_t = b(Y_t) dt + \sigma(Y_t) dZ_t + \int_{\mathbb{R}^n} \zeta(Y_t - x) \tilde{N}(dt, dx), \quad Y_0 = y \in \mathbb{R}^n$$
(B1)
where $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $\zeta: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times l}$ are functions such that a unique solution $Y_t$ exists. Let
$$\tau_\mathcal{F} = \tau_\mathcal{F}(z, \omega) = \inf\{t > 0: Y_t \in \mathcal{F}\}$$
be the bankruptcy time and let $\mathcal{F}$ denote the set of all stopping times $\tau \leq \tau_\mathcal{F}$. Let $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}$ be continuous functions satisfying the conditions
$$E^\gamma[f^-(Y_t) dt] < \infty \quad \text{for all} \ y \in \mathbb{R}^n$$
and the family
\[ \{ g^- (Y_t) \cdot \mathcal{X}_{[\tau < \infty]} : \tau \in \mathcal{T} \} \]
is uniformly integrable for all \( y \in \mathbb{R}^n \). The general optimal stopping problem is the following:

Find \( V(y) \) and \( \tau^* \in \mathcal{T} \) such that

\[ V(y) = \sup_{\tau \in \mathcal{T}} J_\tau (y) = J_{\tau^*} (y), \quad y \in \mathbb{R}^n \]

where

\[ J_\tau (y) = \mathbf{E}^{y} \left[ \int_0^\tau f(Y_t) + g(Y_t) \cdot \mathcal{X}_{[\tau < \infty]} \right], \quad \tau \in \mathcal{T} \]
is the performance criterion.

The function \( V \) is called the value function and the stopping time \( \tau^* \) (if it exists) is called an optimal stopping time. In the sequel, we let \( G \) be the integro-differential operator that coincides with the generator of \( Y_t \) on \( C^2_0 (\mathbb{R}^n) \) given by

\[
G \varphi (y) = \sum_{j=1}^n b_j (y) \frac{\partial \varphi}{\partial y_j} (y) + \frac{1}{2} \sum_{j,k=1}^n (\sigma \sigma^T)_{jk} (y) \frac{\partial^2 \varphi}{\partial y_j \partial y_k} (y) + \int_{\mathbb{R}} \{ \varphi (y + \zeta^{(k)} (y, x_k)) - \varphi (y) - \nabla \varphi (y) \cdot \zeta^{(k)} (y, x_k) \} v_k (dx_k) \tag{B2}
\]
for all \( \varphi : \mathbb{R}^n \to \mathbb{R} \) and \( y \in \mathbb{R}^n \) such that (B2) exists.

**B.1. Integro-variational inequalities for optimal stopping**

Suppose that there exists a function \( \varphi : \overline{\mathcal{T}} \to \mathbb{R} \) such that

(B.1.1) \( \varphi \in C^1 (\mathcal{T}) \cap C (\overline{\mathcal{T}}) \), \( \varphi \geq g \) on \( \mathcal{T} \)

Define the continuation region by

\[ \mathcal{D} = \{ y \in \mathcal{T} : \varphi (y) > g (y) \} \]

and suppose that

(B.1.2) \( \mathbf{E}^{y} \left[ \int_0^\tau \mathcal{X}_{\mathcal{D}^c} Y_t \, dt \right] = 0 \), \( \partial \mathcal{D} \) is a Lipschitz surface

(B.1.3) \( \varphi \in C^2 (\mathcal{T} \setminus \partial \mathcal{D}) \) with locally bounded derivatives near \( \partial \mathcal{D} \), \( G \varphi + f \leq 0 \) on \( \mathcal{T} \setminus \partial \mathcal{D} \)

(B.1.4) \( Y_{\tau^*} \in \partial \mathcal{T} \) a.s. on \( \{ \tau^* < \infty \} \) and

\[
\lim_{t \to \tau^*} \varphi (Y_t) = g (Y_{\tau^*}) \cdot \mathcal{X}_{[\tau^* < \infty]} \]

and

(B.1.5) for all \( \tau \in \mathcal{T} \) we have that

\[
\mathbf{E} \left[ |\varphi (Y_\tau)| + \int_0^{\tau^*} \left\{ |G \varphi (Y_t)| + \sigma^T (Y_t) \nabla \varphi (Y_t) \right\}^2 \right. \\
\left. \times \sum_{k=1}^l \int_{\mathbb{R}} \{ \varphi (Y_t) + \zeta^{(k)} (Y_t, x) - \varphi (Y_t) \}^2 v_k (dx_k) \right] \, dt < \infty \tag{B3}
\]
If (B.1.1)–(B.1.5) hold, then, for all \( y \in \mathcal{F} \), we have that
\[
\varphi(y) \geq V(y)
\]
Moreover, assume that
\[
\text{(B.1.6)} \quad G\varphi + f = 0 \text{ on } \mathcal{D}, \quad \tau_{\mathcal{D}} := \inf\{t > 0 : Y_t \notin \mathcal{D}\} < \infty \text{ a.s for all } y
\]
\[
\text{(B.1.7)} \quad \{\varphi(Y_\tau) : \tau \in \mathcal{F}\} \text{ is uniformly integrable for all } y.
\]
Then we have that
\[
\varphi(y) = V(y) \quad \text{and} \quad \tau^* = \tau_{\mathcal{D}}
\]
is an optimal stopping time.

**B.2. Optimal stopping conditions**

Suppose that the conditions of Appendix A.1 hold. Suppose \( g \in C^2(\mathbb{R}^n) \) and that \( \varphi = g \) satisfies (B3). Define
\[
\mathcal{U} = \{y \in \mathcal{F} : Gg(y) + f(y) > 0\}
\]
Suppose that for all \( y \in \mathcal{U} \) there exists a neighbourhood \( N_y \) of \( y \) such that
\[
\tau_{N_y} := \inf\{t > 0 : Y_t \notin N_y\} < \infty \text{ a.s}
\]
Then
\[
\mathcal{U} \subset \{y \in \mathcal{F} : V(y) > g(y)\} = \mathcal{D}
\]
Hence, it is never optimal to stop while \( Y_t \in \mathcal{U} \).

**B.3. Proof of Theorem 3.5**

We can show that for the choices of \( z^* \) and \( K^{(1)} \) made in (56) and \( \varphi \) given by (57) all the conditions given in Appendices B.1 (conditions (B.1.1)–(B.1.7) particularly) and B.2 are satisfied.

**APPENDIX C: PROOF OF THEOREM 3.3**

The proof is reliant on [1]. Condition (52) holds if and only if
\[
(\varepsilon^{(1)} = \varepsilon^{(2)} ) \quad \text{(C1)}
\]
This equation is needed to fix \( A^s \) and \( A^r \) endogenously. By making use of (26) and (36), the condition (C1) becomes
\[
\frac{f A^r}{m+f} + (\varepsilon^{(1)} - \varepsilon^{(2)})q_1 A^r q_1 + (\varepsilon^{(2)} - \varepsilon^{(2)})q_2 A^r q_2 = 0 \quad \text{(C2)}
\]
In addition, it is possible to demonstrate that $A^r$ and $A^s$ are the solutions to the non-linear system of equations

\[
\begin{align*}
[\tilde{e}^{(1)}(A^r, A^s) - \varepsilon^{(1)}(A^r, A^s)]q_1 A^rq_1 + [\tilde{e}^{(2)}(A^r, A^s) - \varepsilon^{(2)}(A^r, A^s)]q_2 A^sq_2 - \frac{f A^r}{m + f} &= 0 \quad \text{(C3)} \\
[\tilde{e}^{(1)}(A^r, A^s) - \varepsilon^{(1)}(A^r, A^s)]q_1 A^sq_1 + [\tilde{e}^{(2)}(A^r, A^s) - \varepsilon^{(2)}(A^r, A^s)]q_2 A^rq_2 + \frac{f A^s}{m + f} &= 0 \quad \text{(C4)}
\end{align*}
\]

We have to prove the existence of a solution for (C3) and (C4) with the property that $A^s \geq A^r \geq 0$. In order to accomplish this, the substitution of $u = A^r / A^s$ makes matters easier. For the ratio $u$ defined above, (C3) and (C4) becomes

\[
\begin{align*}
A^s &\left[ fp_1(q_1 - q_2)u^{q_1 - q_2} - mp_1u[(1 - q_1)u^{q_1} - (1 - q_2)u^{q_2}] \right] \\
&\quad + \frac{c^+(\tau - 1)[p_1(q_1 - q_2)u^{q_1} + (m/r^T)p_1(q_1u^{q_1} - q_2u^{q_2})]}{m + r^T} = 0 \quad \text{(C5)} \\
A^r &\left[ f[(p_1 - q_1)(1 - q_1)u^{q_1} - (p_1 - q_2)(1 - q_1)u^{q_2}] - mu(1 - p_1)(q_1 - q_2) \right] \\
&\quad + \frac{c^+(\tau - 1)[q_1(p_1 - q_2)u^{q_1} - q_2(p_1 - q_1)u^{q_1} + (m/r^T)p_1(q_1 - q_2)]}{m + r^T} = 0 \quad \text{(C6)}
\end{align*}
\]

It is clear that (C5) and (C6) are linear in $A^s$ and non-linear in $u$. Therefore, following [1], we may express $A^s$ as a function of $u$ and decouple $u$ from $A^s$. This procedure yields

\[
A^s = \frac{c^+(1 - \tau)(m + f)[p_1(q_1 - q_2)u^{q_1} + (m/r^T)p_1(q_1u^{q_1} - q_2u^{q_2})]}{(m + r^T)[fp_1(q_1 - q_2)u^{q_1} - q_2(p_1 - q_1)u^{q_1} - mp_1u[(1 - q_1)u^{q_1} - (1 - q_2)u^{q_2}]}} \quad \text{(C7)}
\]

As a consequence, only a nonlinear equation in $u$ given by (53) remains. Furthermore, we are able to show that the right-hand side of (C7) is always positive and thus $A^s$ is always positive. Finally, it is clear that (53) does not depend on $c^+$ and $\tau$ and, therefore, also $u$ is independent of these parameters.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the insightful comments made by two anonymous referees and the subject editor, Prof. Mahmut Parlar. Their suggestions have contributed to a much improved version of the paper. We are also indebted to the National Research Foundation (GUN No. 2074218) for financial support.

REFERENCES


